

# SOLITONIC MODELS BASED ON QUANTUM GROUPS AND THE STANDARD MODEL

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**Abstract.** The idea that the elementary particles might have the symmetry of knots has had a long history. In any current formulation of this idea, however, the knot must be quantized. The present review is a summary of a small set of papers that began as an attempt to correlate the properties of quantized knots with the empirical properties of the elementary particles. As the ideas behind these papers have developed over a number of years the model has evolved, and this review is intended to present the model in its current form. The original picture of an elementary fermion as a solitonic knot of field, described by the trefoil representation of  $SU_q(2)$ , has expanded into its current form in which a knotted field is complementary to a composite structure composed of three or more preons that in turn are described by the fundamental representation of  $SL_q(2)$ . These complementary descriptions may be interpreted as describing single composite particles composed of three or more preons bound by a knotted field.

# 1 Introduction

The possibility that the elementary particles are knots has been suggested by many authors, going back as far as Kelvin.<sup>1</sup> Among the different field theoretic attempts to construct classical knots, a model related to the Skyrme soliton has been described by Fadeev and Niemi.<sup>2</sup> There are also the familiar knots of a magnetic field; and since these are macroscopic expressions of the electroweak field, it is natural to extrapolate from macroscopic to microscopic knots of this same field. One expects that the conjectured microscopic knots would be quantized, and that they would be observed as solitonic in virtue of both their topological and quantum stability. It is then natural to ask if the elementary particles might also be knots. If they are, one expects that the most elementary particles, namely the fermions, are also the most elementary knots, namely the trefoils. This possibility is suggested by the fact that there are 4 quantum trefoils and 4 classes of elementary fermions, and is supported by a unique one-to-one correspondence between the topological description of the 4 quantum trefoils and the quantum numbers of the 4 fermionic classes. We have first attempted to determine the minimal restrictions on a model of the elementary particles in the context of weak interactions if the knotted soliton (quantum knot) is described only by its symmetry algebra  $SL_q(2)$  independent of its field theoretic origin. The use of this symmetry algebra to define the quantum knot is similar to the use of the symmetry algebra of the rotation group to define the quantum spin. Before describing the symmetry algebra  $SL_q(2)$  we shall describe an oriented knot by its topological invariants and by an invariant polynomial.

## 2 The Characterization of Oriented Knots

Three-dimensional knots are described in terms of their projections onto a two-dimensional plane where they appear as two-dimensional curves with 4-valent vertices. At each vertex (crossing) there is an overline and an underline. We shall be interested here in oriented knots. The crossing sign of the vertex is +1 or -1 depending on whether the orientation of the overline is carried into the orientation of the underline by a counterclockwise or clockwise rotation, respectively. The sum of all crossing signs is termed the writhe,  $w$ , a topological

invariant. There is a second topological invariant, the rotation,  $r$ , the number of rotations of the tangent in going once around the knot.

Let  $K$  and  $K'$  be oriented knot diagrams with the same writhe and rotation

$$\begin{aligned} w(K) &= w(K') \\ r(K) &= r(K') \end{aligned}$$

Then  $K$  is topologically equivalent (regularly isotopic) to  $K'$ .

We may label an oriented knot by the number of crossings ( $N$ ), its writhe ( $w$ ), and rotation ( $r$ ). The writhe and rotation are integers of opposite parity.

### 3 The Kauffman Algorithm for Associating a Polynomial with a Knot<sup>3</sup>

Denote the Kauffman polynomial associated with a knot,  $K$ , having  $n$  crossings, by  $\langle K \rangle_n$ . Let us represent  $\langle K \rangle_n$  by the bracket

$$\langle K \rangle_n \sim \left\langle \begin{array}{c} \cdots \\ \times \end{array} \right\rangle \quad (3.1)$$

The interior of this bracket is intended to represent the projected knot when only one of the  $n$  crossings is explicitly shown. Let us also introduce the polynomials  $\langle K_{\pm} \rangle_{n-1}$ , associated with slightly altered diagrams in which the crossing lines are reconnected, as follows:

$$\langle K_- \rangle_{n-1} \sim \left\langle \begin{array}{c} \cdots \\ \asymp \end{array} \right\rangle \quad \text{and} \quad \langle K_+ \rangle_{n-1} \sim \left\langle \begin{array}{c} \cdots \\ )( \end{array} \right\rangle \quad (3.2)$$

Then one may define a Laurent polynomial in the parameter  $q$  by the following recursive rules:

$$\langle K \rangle_n = i \left[ q^{-1/2} \langle K_- \rangle_{n-1} - q^{1/2} \langle K_+ \rangle_{n-1} \right] \quad (I)$$

$$\langle OK \rangle = (q + q^{-1}) \langle K \rangle \quad (II)$$

$$\langle O \rangle = q + q^{-1} \quad (III)$$

Every time Rule I is applied, one crossing is eliminated and the number of diagrams is doubled. After  $n$  applications of Rule I, we find  $2^n$  diagrams, each with one or more internal

loops. When these loops, indicated by  $O$ , are removed by Rules II and III we are left with a Laurent polynomial in  $q$ . Then  $\langle K \rangle_n$  is the Kauffman polynomial associated with the knot with  $n$  crossings denoted by  $K_n$ .

The Kauffman rules may be written entirely in terms of the Pauli matrices  $\sigma_{\pm}$  and the following matrix:

$$\epsilon_q = \begin{pmatrix} 0 & q_1^{1/2} \\ -q_1^{1/2} & 0 \end{pmatrix} \quad q_1 = q^{-1} \quad (3.3)$$

These rules then read as follows:

$$\langle K \rangle_n = \text{Tr } \epsilon_q [\sigma_- \langle K_- \rangle_{n-1} + \sigma_+ \langle K_+ \rangle_{n-1}] \quad (I)'$$

$$\langle OK \rangle = \text{Tr } \epsilon_q^t \epsilon_q \langle K \rangle \quad (II)'$$

$$\langle O \rangle = \text{Tr } \epsilon_q^t \epsilon_q \quad (III)'$$

where

$$\sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i \sigma_2)$$

and the  $\vec{\sigma}$  are the Pauli matrices.

One may obtain an invariant of ambient isotopy by forming<sup>3,4</sup>

$$f_K(A) = (-A^3)^{-w(K)} \langle K \rangle \quad (3.4)$$

where  $w(K)$  is the writhe of  $K$  and

$$A = i \text{Tr } \epsilon_q \sigma_- \quad (3.5)$$

The Jones polynomial is

$$V_K(t) = f_K(t^{-1/4}) \quad (3.6)$$

The Kauffman and Jones polynomials are topological invariants. They are invariants of regular and ambient isotopy respectively.

## 4 The Knot Algebra<sup>4,5,6</sup>

The description of the knot by  $(I)'$ ,  $(II)'$ ,  $(III)'$  is invariant under the transformations

$$\epsilon'_q = T \epsilon_q T^t = T^t \epsilon_q T \quad (4.1a)$$

$$\vec{\sigma}' = \vec{\sigma} \quad (4.1b)$$

where

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (4.2)$$

if the matrix elements of  $T$  satisfy the following algebra:

$$\begin{aligned} ab &= qba & bd &= qdb & ad - qbc &= 1 & bc &= cb \\ ac &= qca & cd &= qdc & da - q_1cb &= 1 \end{aligned} \quad (A)$$

Then

$$T\epsilon_q T^t = T^t \epsilon_q T = \epsilon_q \quad (4.3)$$

and by (4.1a)

$$\epsilon'_q = \epsilon_q \quad (4.4)$$

Therefore the Kauffman algorithm as expressed in terms of  $\epsilon_q$  is invariant under (4.1). We shall refer to (A) as the knot algebra. The matrix  $T$ , as defined by (4.2) and (A), is a 2-dimensional representation of  $SL_q(2)$ .

We shall also introduce the unitary algebra  $SU_q(2)$  obtained by setting

$$\begin{aligned} d &= \bar{a} \\ c &= -q_1 \bar{b} \end{aligned} \quad (4.5)$$

Then (A) reduces to

$$\begin{aligned} ab &= qba & a\bar{a} + b\bar{b} &= 1 & b\bar{b} &= \bar{b}b \\ a\bar{b} &= q\bar{b}a & \bar{a}a + q_1^2 \bar{b}b &= 1 \end{aligned} \quad (A)'$$

The Kauffmann and Jones knot polynomials are left invariant by (4.1) subject to either (A) or (A)', the algebras defining the two-dimensional representation of  $SL_q(2)$  and  $SU_q(2)$ . For the physical applications we need the higher dimensional representations of  $SL_q(2)$  and  $SU_q(2)$ .

## 5 Higher Dimensional Representations of $SL_q(2)$ and $SU_q(2)$

To compute the higher dimensional representations one needs the  $q$ -binomial theorem.<sup>7</sup> This may be written in either of the following two ways:

$$(A + B)^n = \sum \left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_q B^s A^{n-s} \quad (5.1a)$$

or as

$$(A + B)^n = \sum \left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_{q_1} A^s B^{n-s} \quad (5.1b)$$

where

$$AB = qBA \quad \text{and} \quad q_1 = q^{-1} \quad (5.2)$$

Here

$$\left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_q = \frac{\langle n \rangle_q!}{\langle n-s \rangle_q! \langle s \rangle_q!} \quad \text{with} \quad \langle n \rangle_q = \frac{q^n - 1}{q - 1} \quad (5.3)$$

We shall use this theorem to compute the transformations on the following class of monomials:

$$\psi_m^j = N_m^j x_1^{n_+} x_2^{n_-} \quad -j \leq m \leq j \quad (5.4a)$$

where

$$[x_1, x_2] = 0 \quad (5.4b)$$

$$n_{\pm} = j \pm m \quad (5.4c)$$

$$N_m^j = \frac{1}{[\langle n_+ \rangle_{q_1}! \langle n_- \rangle_{q_1}!]^{1/2}} \quad (5.4d)$$

when  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$  is transformed according to the 2-dimensional representations of  $SL_q(2)$  as follows:

$$x'_1 = ax_1 + bx_2 \quad (5.5)$$

$$x'_2 = cx_1 + dx_2 \quad (5.6)$$

Here  $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  is the 2-dimensional representation of  $SL_q(2)$  introduced in (4.2).

Then

$$\psi_m^{j'} = N_m^j (ax_1 + bx_2)^{n_+} (cx_1 + dx_2)^{n_-} \quad (5.7)$$

We assume that  $(a, b, c, d)$  commute with  $(x_1, x_2)$  so that

$$(ax_1)(bx_2) = q(bx_2)(ax_1) \quad (5.8)$$

$$(cx_1)(dx_2) = q(dx_2)(cx_1) \quad (5.9)$$

By the  $q$ -binomial theorem

$$\psi_m^{j'} = N_m^j \sum_s \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} (ax_1)^s (bx_2)^{n_+ - s} \sum_t \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} (cx_1)^t (dx_2)^{n_- - t} \quad (5.10)$$

$$\begin{aligned} &= N_m^j \sum_{s,t} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} x_1^{s+t} x_2^{n_+ + n_- - s - t} a^s b^{n_+ - s} c^t d^{n_- - t} \\ &= N_m^j \sum_{s,t} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} a^s b^{n_+ - s} c^t d^{n_- - t} x_1^{n'_+} x_2^{n'_-} \end{aligned} \quad (5.11)$$

where

$$n'_+ = s + t \quad (5.12)$$

$$n'_- = n_+ + n_- - s - t \quad (5.13)$$

and by (5.4c)

$$n'_+ + n'_- = n_+ + n_- = 2j \quad (5.14)$$

Set

$$n'_\pm = j \pm m' \quad (5.15)$$

We may rewrite (5.11) as

$$\psi_m^{j'} = \sum_{s,t} \left( \frac{N_m^j}{N_{m'}^j} \right) \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} \delta(s + t, n'_+) a^s b^{n_+ - s} c^t d^{n_- - t} (N_{m'}^j x_1^{n'_+} x_2^{n'_-}) \quad (5.16)$$

$$= \sum_{m'} D_{mm'}^j \psi_{m'}^j \quad (5.17)$$

where

$$D_{mm'}^j = \frac{N_m^j}{N_{m'}^j} \sum_{s,t} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} \delta(s + t, n'_+) a^s b^{n_+ - s} c^t d^{n_- - t} \quad (5.18)$$

or

$$D_{mm'}^j = \left( \frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right)^{1/2} \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_1 \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_1 \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \quad (5.19)$$

where we write  $\langle \rangle_1$  for  $\langle \rangle_{q_1}$ . The corresponding representations of  $SU_q(2)$  are obtained by setting

$$d = \bar{a} \quad (5.20a)$$

$$c = -q_1 \bar{b} \quad (5.20b)$$

Then

$$D_{mm'}^j = \left( \frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right)^{1/2} \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_1 \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_1 \delta(s+t, n'_+) (-q_1)^t a^s b^{n_+-s} \bar{b}^t \bar{a}^{n_--t} \quad (5.21)$$

For both  $SL_q(2)$  and  $SU_q(2)$  we have

$$\psi_m^j(x'_1, x'_2) = \sum D_{mm'}^j \psi_{m'}^j(x_1, x_2) \quad (5.22)$$

In obtaining these representations of  $SL_q(2)$  and  $SU_q(2)$  that operate on the monomial basis (5.4a) we are following a well known procedure for obtaining representations of  $SU(2)$ .<sup>8</sup>

## 6 The Gauge Group of the $SL_q(2)$ and $SU_q(2)$ Algebras

By (5.21) the  $2j+1$ -dimensional representations of  $SL_q(2)$  have the following form

$$D_{mm'}^j = \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \mathcal{A}_{mm'}^j(q, s, t) \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \quad (6.1)$$

where  $(a, b, c, d)$  satisfy the knot algebra  $(A)$  defined in Section 4. Here

$$n_{\pm} = j \pm m \quad (6.2)$$

$$n'_{\pm} = j \pm m' \quad (6.3)$$

$D_{mm'}^j$  is defined only up to the following gauge transformation on  $(a, b, c, d)$  that leaves the algebra  $(A)$  invariant:



$$\begin{aligned}
a' &= e^{i\varphi_a} a & b' &= e^{i\varphi_b} b \\
d' &= e^{-i\varphi_a} d & c' &= e^{-i\varphi_b} c
\end{aligned} \tag{G}$$

We shall also refer to (G) as  $U_a(1) \times U_b(1)$ . Under the gauge transformation (G), every term in  $D_{mm'}^j$  transforms like

$$(a^{n_a} b^{n_b} c^{n_c} d^{n_d})' = e^{i\varphi_a(n_a - n_d)} e^{i\varphi_b(n_b - n_c)} (a^{n_a} b^{n_b} c^{n_c} d^{n_d}) \tag{6.4}$$

But by the  $\delta$ -function in (6.1)

$$\begin{aligned}
n_a - n_d &= s + (t - n_-) = n'_+ - n_- = m' + m \\
n_b - n_c &= (n_+ - s) - t = n_+ - n'_+ = m - m'
\end{aligned} \tag{6.5}$$

By (6.4) and (6.5) every term of  $D_{mm'}^j$  transforms the same way and therefore the  $D_{mm'}^j$  transforms under  $G$  as follows:

$$D_{mm'}^{j \prime} = e^{i(m+m')\varphi_a} e^{i(m-m')\varphi_b} D_{mm'}^j \tag{6.6a}$$

or

$$D_{mm'}^{j \prime} = e^{i(\varphi_a + \varphi_b)m} e^{i(\varphi_a - \varphi_b)m'} D_{mm'}^j \tag{6.6b}$$

We denote the irreducible representations of  $SU_q(2)$  by  $D_{mm'}^j(a, \bar{a}, b, \bar{b})$ .

The gauge transformations on  $SU_q(2)$ , namely

$$\begin{aligned}
a' &= e^{i\varphi_a} a \\
b' &= e^{i\varphi_b} b
\end{aligned} \tag{6.7}$$

induce the same transformations (6.6) on the  $D_{mm'}^j(a, \bar{a}, b, \bar{b})$ .

## 7 Representation of an Oriented Knot

The oriented knot has three coordinates, namely  $(N, w, r)$  the number of crossings  $N$ , the writhe  $w$ , and the rotation  $r$ . We may make a coordinate transformation to  $(j, m, m')$ , the indices that label the irreducible representations  $D_{mm'}^j$  of  $SL_q(2)$  by setting

$$\begin{aligned}
j &= N/2 \\
m &= w/2 \\
m' &= (r + 1)/2
\end{aligned} \tag{7.1}$$

This linear transformation allows half-integer representations and respects the knot constraint requiring  $w$  and  $r$  to be of opposite parity. In this new coordinate system one may label the knot  $(N, w, r)$  by  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}(a, b, c, d)$ . One thereby associates with the  $(N, w, r)$  knot a multinomial in the elements of the algebra of the form

$$D_{mm'}^j(abcd) = \sum \mathcal{A}_{mm'}^j(q, s, t) \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \quad (7.2)$$

where explicit forms of  $\mathcal{A}_{mm}^j$  are given in (5.19) and (5.21).

Like the Kauffman and Jones polynomials these forms are based on the algebra of the classical knot. They are operator expressions that may be evaluated on the state space of the algebra.

Let us next compute a basis in this space.

Since  $b$  and  $c$  commute, they have common eigenstates. Let  $|0\rangle$  be designated as a ground state and let

$$b|0\rangle = \beta|0\rangle \quad (7.3)$$

$$c|0\rangle = \gamma|0\rangle \quad (7.4)$$

$$bc|0\rangle = \beta\gamma|0\rangle \quad (7.5)$$

We may assume that  $b$  and  $c$  are Hermitian:

$$b = \bar{b} \quad (7.6)$$

$$c = \bar{c} \quad (7.7)$$

Then the eigenvalues  $\beta, \gamma$  are real and the eigenfunctions are orthogonal.

From the algebra we see that

$$bc|n\rangle = E_n|n\rangle \quad (7.8)$$

where

$$|n\rangle \sim d^n|0\rangle \quad (7.9)$$

and

$$E_n = q^{2n} \beta \gamma \quad (7.10)$$

This eigenvalue spectrum resembles that of a harmonic oscillator but the levels are arranged in geometrical rather than arithmetical progression. We shall refer to this spectrum as the  $q$ -oscillator spectrum.

Here  $d$  and  $a$  are raising and lowering operators respectively.

$$d|n\rangle = \lambda_n|n+1\rangle \quad (7.11)$$

$$a|n\rangle = \mu_n|n-1\rangle \quad (7.12)$$

If there is a highest state  $M$ ,  $\lambda_M = 0$ ; if there is a lowest state  $M'$ ,  $\mu_{M'} = 0$ . We also have

$$\begin{aligned} ad|n\rangle &= a\lambda_n|n+1\rangle \\ &= \lambda_n\mu_{n+1}|n\rangle \end{aligned} \quad (7.13)$$

$$\begin{aligned} da|n\rangle &= d\mu_n|n-1\rangle \\ &= \mu_n\lambda_{n-1}|n\rangle \end{aligned} \quad (7.14)$$

From the algebra (A), (7.13) and (7.14) become

$$(1 + qbc)|n\rangle = \lambda_n\mu_{n+1}|n\rangle \quad (7.15)$$

$$(1 + q_1bc)|n\rangle = \mu_n\lambda_{n-1}|n\rangle \quad (7.16)$$

If there is both a highest state  $M$ , and a lowest state  $M'$ , then

$$\lambda_M = \mu_{M'} = 0 \quad M' < M \quad (7.17)$$

and by (7.15) and (7.16)

$$(1 + qbc)|M\rangle = 0 \quad (7.18)$$

$$(1 + q_1bc)|M'\rangle = 0 \quad (7.19)$$

Then by (7.8) and (7.10)

$$q^{2M+1}\beta\gamma = q^{2M'-1}\beta\gamma \quad (7.20)$$

or

$$(q^2)^{M-M'+1} = 1 \quad (7.21)$$

We assume that  $q$  is real, so that

$$M' = M + 1 \quad (7.22)$$

Since (7.17) and (7.22) are contradictory, there may be either a highest or a lowest state but not both. The same discussion may be given for the  $SU_q(2)$  algebra.

In the next section we shall assume that the individual states of excitation of the quantum knot are to be represented by  $D_{mm'}^j|n\rangle$ . Since the empirical evidence appears to restrict the number of states, there must be an externally required physical boundary condition to cut off the otherwise infinite spectrum that is formally required by (7.10).

## 8 The Quantum Knot<sup>9,10,11,12</sup>

Since the writhe and rotation of a classical knot are regular topological invariants, they do not depend on the size or shape of the knot; i.e., they are conformal invariants that hold for microscopic knots as well. It follows that  $w$  and  $r$  are integrals of the motion for microscopic classical knots with spectra determined by the topology of the knot.

We shall now introduce the quantum knot by interpreting  $D_{mm'}^j(a, b, c, d)$  as the kinematical description of a quantum state, where

$$(j, m, m') = \frac{1}{2}(N, w, \pm r + 1) \quad (8.1)$$

and  $(N, w, r)$  describes a classical knot. Since the spectra of  $(j, m, m')$  are restricted by  $SL_q(2)$ , and the spectra of  $(N, w, r)$  are restricted by knot topology, the states of the quantized knot are thus jointly restricted by both  $SL_q(2)$  and the knot topology. The equations (8.1) establish a correspondence between a quantized knot described by  $D_{\frac{w}{2}, \frac{\pm r + 1}{2}}^{N/2}$  and a classical knot described by  $(N, w, r)$ , but the correspondence is not one-to-one. There is a one-to-one correspondence between the quantum trefoil and the 2d-projection.

For the trefoil configuration there are four choices of  $(w, r)$ , namely  $(3, 2)$ ,  $(-3, 2)$ ,  $(3, -2)$ ,  $(-3, -2)$ . Regarded as classical knots, only two of these trefoils are topologically distinct; but we shall consider all four choices of  $D_{\frac{w}{2}, \frac{\pm r + 1}{2}}^{3/2}$  as distinct quantum states, since the “topological degeneracy” is lifted by turning on the hypercharge, as we shall see. In the following when  $(w, r)$  refers to the quantum knot, both  $w$  and  $r$  may have either sign.

One may similarly define the eigenstates of the spherical top as irreducible representations of  $O(3)$  by  $D_{mm'}^j(\alpha, \beta, \gamma)$  where the indices  $(j, m, m')$  refer to the angular momen-

tum of the top and the arguments  $(\alpha, \beta, \gamma)$  to its orientation. It is also possible to define the eigenstates of the hydrogen atom as irreducible representations of  $O(3)$ , expressed as  $D_{mm'}^j(a_1, a_2, a_3)$  where in this case  $(a_1, a_2, a_3)$  are three coordinates on the group space of  $O(3)$ , and where  $(2j + 1, m, m')$  are respectively the principal quantum number, the  $z$ -component of the angular momentum, and the  $z$ -component of the Runge-Lenz vector.<sup>9</sup> Here the quantum knot is similarly described, but it is defined on the  $SL_q(2)$  algebra, a discrete space rather than a three-dimensional continuum.

If the knot oscillates like a quantum mechanical harmonic oscillator, the Hamiltonian is of the following form:

$$H = (a\bar{a} + \bar{a}a)\frac{\hbar\omega}{2} \quad (8.2a)$$

where  $\bar{a}$  and  $a$  are raising and lowering operators and

$$[a, \bar{a}] = 1 \quad (8.2b)$$

Since the raising and lowering operators of the  $SL_q(2)$  algebra that correspond to  $\bar{a}$  and  $a$  of the harmonic oscillator are  $d$  and  $a$ , the knot analogue of (8.2) is

$$H = (ad + da)\frac{\hbar\omega}{2} \quad (8.3a)$$

but

$$[a, d] = (q - q_1)bc \quad (8.3b)$$

and

$$\frac{1}{2}(ad + da) = 1 + \frac{1}{2}(q + q_1)bc \quad (8.4)$$

We may generalize the  $SL_q(2)$  analog of the harmonic oscillator by replacing (8.3) by a more general function of  $ad + da$ , namely:

$$H = H(ad + da)\frac{\hbar\omega}{2} \quad (8.5)$$

or by (8.4) with a different  $H$

$$H = H(bc)\frac{\hbar\omega}{2} \quad (8.6)$$

or with a still different  $H$

$$H = H(b, c)\frac{\hbar\omega}{2} \quad (8.7)$$

Let the Hamiltonian of a quantum knot be  $H(b, c)$ . Let us consider the states of this knot defined by  $D_{mm'}^j |n\rangle$ . We may then compute

$$H(b, c) D_{mm'}^j |n\rangle = H(b, c) \left[ \sum_{s,t} \mathcal{A}_{mm'}^j \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \right] |n\rangle \quad (8.8)$$

$$= D_{mm'}^j H(q_1^{n_a-n_d} b, q_1^{n_a-n_d} c) |n\rangle \quad (8.9)$$

where  $n_a$  and  $n_d$  are the exponents of  $a$  and  $d$  respectively, and  $n_a - n_d$  is by (6.5) the same for every term of  $D_{mm'}^j$ . Then one has by (7.3), (7.4), and (7.10)

$$\begin{aligned} H(b, c) D_{mm'}^j |n\rangle &= D_{mm'}^j H(q_1^{n_a-n_d} q^n \beta, q_1^{n_a-n_d} q^n \gamma) |n\rangle \\ &= E_{mm'}^j(n) \mathcal{D}_{mm'}^j |n\rangle \end{aligned} \quad (8.10)$$

where the eigenvalues of  $H$  are

$$E_{mm'}^j(n) = H(\lambda\beta, \lambda\gamma) \quad (8.11)$$

and

$$\lambda = q^{n-(m+m')} \quad (8.12)$$

by (6.5). The eigenstates of  $H$  are the  $D_{mm'}^j |n\rangle$  and the indices on  $D_{mm'}^j$  are the eigenvalues of the integrals of motion.

The operators that represent the integrals of the motion may be expressed in terms of an elementary operator  $\omega_x$  that may be defined by its action on every term of  $D_{mm'}^j$  as follows:

$$\omega_x(\dots x^{n_x} \dots) = n_x(\dots x^{n_x} \dots) \quad x = (a, b, c, d) \quad (8.13)$$

i.e.,  $\omega_x$  acts like  $x \frac{\partial}{\partial x}$ , a dilatation operator.

Then define

$$\mathcal{N} = (\omega_a + \omega_b + \omega_c + \omega_d) \quad (8.14)$$

$$\mathcal{W} = (\omega_a - \omega_d + \omega_b - \omega_c) \quad (8.15)$$

$$\mathcal{R} = (\omega_a - \omega_d - \omega_b + \omega_c) \quad (8.16)$$

When  $\mathcal{N}$ ,  $\mathcal{W}$ , and  $\mathcal{R}$  act on  $D_{mm'}^j$  one finds by (6.1)

$$\mathcal{N} D_{mm'}^j = 2j D_{mm'}^j \quad (8.17)$$

$$\mathcal{W} D_{mm'}^j = 2m D_{mm'}^j \quad (8.18)$$

$$\mathcal{R} D_{mm'}^j = 2m' D_{mm'}^j \quad (8.19)$$

We shall describe a state function of the quantum knot by

$$\psi_{wr}^N = D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} |n\rangle \quad (8.20)$$

following (7.1) where  $(N, w, |r|)$  are the number of crossings, writhe, and rotation of the corresponding classical knot. Then by Eqs. (8.17)-(8.19) we have

$$\mathcal{N} \psi_{wr}^N = N \psi_{wr}^N \quad (8.21)$$

$$\mathcal{W} \psi_{wr}^N = w \psi_{wr}^N \quad (8.22)$$

$$\mathcal{R} \psi_{wr}^N = (r+1) \psi_{wr}^N \quad (8.23)$$

where the spectra of  $(\mathcal{N}, \mathcal{W}, \mathcal{R})$  are restricted by the topology of the knot. In addition we have

$$H(b, c) \psi_{wr}^N = E_{wr}^N \psi_{wr}^N \quad (8.24)$$

where by (8.12) and (7.1)

$$E_{wr}^N = H(\lambda\beta, \lambda\gamma) \quad \text{and} \quad (8.25a)$$

$$\lambda = q^{n - \frac{1}{2}(w+r+1)} \quad (8.25b)$$

## 9 The Quantum Knot and the Standard Theory<sup>10,11,12</sup>

One may now attempt to relate  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}(q|abcd)$  to the internal state of an elementary particle, which we shall assume to be a boson if  $N$  is even and a fermion if  $N$  is odd. Since the lowest value of  $N$  is 3, corresponding to a trefoil, we shall try to identify the four quantum trefoils with the four classes of elementary fermions, namely

$$(1) \quad \nu_e \quad \nu_\mu \quad \nu_\tau$$

$$(2) \quad e \quad \mu \quad \tau$$

$$(3) \quad d \quad s \quad b$$

$$(4) \quad u \quad c \quad t$$

We shall assume that each quantum trefoil has 3 states of excitation, e.g., the 3 states of the leptonic trefoil represent  $(e, \mu, \tau)$ . We shall now represent all four of the elementary fermionic trefoils by  $D_{\frac{w}{2} \frac{r+1}{2}}^{3/2}$  and the three states of each trefoil by  $D_{\frac{w}{2} \frac{r+1}{2}}^{3/2} |n\rangle$ ,  $n = 0, 1, 2$ .

In order to identify the 4 quantum trefoils with the 4 families of fermions, it is necessary to establish a unique correspondence between the 4 choices of writhe and rotation that label the quantum trefoils and the 4 choices of charge and hypercharge that distinguish the 4 families of fermions. For this purpose we introduce two “knot charges”  $Q_a$  and  $Q_b$  by rewriting (6.6) as follows:

Let

$$Q_a \equiv -k(m + m') = -k \frac{w + r + 1}{2} \quad (9.1)$$

$$Q_b \equiv -k(m - m') = -k \frac{w - r - 1}{2} \quad (9.2)$$

where  $k$  is an undetermined constant with the dimensions of an electric charge. The classical  $Q_a$  and  $Q_b$  are conserved since  $w$  and  $r$  are conserved. By (6.6) the gauge transformations ( $G$ ) on the algebra ( $A$ ) induce the following gauge transformations on the kinematical states

$$D_{mm'}^{j'} = U_a U_b D_{mm'}^j \quad (9.3)$$

where

$$U_a = e^{-ik^{-1}Q_a\varphi_a} \quad (9.4)$$

$$U_b = e^{-ik^{-1}Q_b\varphi_b} \quad (9.5)$$

Then  $U_a$  and  $U_b$  are two independent gauge transformations on the knot states, and may be compared with the two independent gauge transformations defining charge and hypercharge. To examine this correspondence we compare the knot charges  $Q_a$  of the 4 quantum trefoils with the electric charges  $Q_f$  of the four families in Table 9.1.

**Table 9.1**

Trefoils ( $w, r$ )	$D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$	$Q_a$	Fermion Class	$Q_f$
$(-3, 2)$	$D_{-\frac{3}{2} \frac{3}{2}}^{3/2}$	0	$(\nu_e \nu_\mu \nu_\tau)$	0
$(3, 2)$	$D_{\frac{3}{2} \frac{3}{2}}^{3/2}$	$-3k$	$(e^-, \mu^-, \tau^-)$	$-e$
$(3, -2)$	$D_{\frac{3}{2} -\frac{1}{2}}^{3/2}$	$-k$	$(d, s, b)$	$-\frac{1}{3}e$
$(-3, -2)$	$D_{-\frac{3}{2} -\frac{1}{2}}^{3/2}$	$2k$	$(u, c, t)$	$\frac{2}{3}e$



In Table (9.1) and by Eq. (9.1)

$$\begin{aligned} Q_a &= -k(m + m') = -\frac{k}{2}(w + r + 1) \\ Q_f &= \text{electric charge of fermion class} \end{aligned} \quad (9.6)$$

There is a unique mapping and single value of  $k$  that permits one to match the trefoil knots with the fermion classes by satisfying

$$Q_a(w, r) = Q_f \quad (9.7)$$

where  $k$  appears as the quantum of charge:

$$k = \frac{e}{3} \quad (9.8)$$

Then

$$Q_a = -\frac{e}{6}(w + r + 1) \quad (9.9)$$

may be considered the electric charge of the quantum trefoil.

The above mapping is unique in the sense that any other correspondence between the trefoils and the fermion classes would destroy the proportionality between  $Q_a$  and  $Q_f$  and would therefore require more than a single value of  $k$ .

Since  $Q_a \sim m + m' = n_a - n_d$  by (9.1) and (6.5), note that the vanishing of  $Q_a$  implies

$$n_a = n_d \quad (9.10)$$

and therefore that  $a$  and  $d$  may be eliminated from every term of  $D_{mm'}^j$  with the aid of

$$a^n d^n = \prod_{s=1}^n (1 + q^{2s-1}bc) \quad (9.11)$$

as follows from the  $SL_q(2)$  algebra ( $A$ ). Therefore electrically neutral states (neutrinos and neutral bosons) lie entirely in the  $(b, c)$  subalgebra.

If the symmetry group is  $SU_q(2)$  then the neutral states lie in the  $(b, \bar{b})$  subalgebra. Also  $\bar{D}_{mm'}^j$  has opposite charges from  $D_{mm'}^j$  and may be identified as the state of the antiparticle.

Given the match in Table (9.1) we may now compare all the quantum numbers  $(t, t_3, Q)$  labeling the different classes of fermions in the standard representation with the quantum numbers  $(N, w, r)$  labeling the corresponding quantum trefoils.

**Table 9.2**

Standard Representation				Trefoil Representation			
	$t$	$t_3$	$Q$	$w$	$r$	$D_{\frac{w}{2}-\frac{r+1}{2}}^{3/2}$	$Q_a$
$(e\mu\tau)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$-e$	3	2	$D_{\frac{3}{2}-\frac{3}{2}}^{3/2}$	$-e$
$(\nu_e\nu_\mu\nu_\tau)_L$	$\frac{1}{2}$	$\frac{1}{2}$	0	-3	2	$D_{-\frac{3}{2}-\frac{3}{2}}^{3/2}$	0
$(dsb)_L$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3} e$	3	-2	$D_{\frac{3}{2}-\frac{1}{2}}^{3/2}$	$-\frac{1}{3} e$
$(uct)_L$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3} e$	-3	-2	$D_{-\frac{3}{2}-\frac{1}{2}}^{3/2}$	$\frac{2}{3} e$

One then reads off the following relations from Table 9.2.

$$t = \frac{N}{6} \quad (9.12)$$

since  $N = 3$  for trefoils.

Also  $t_3$  is proportional to  $w$  (not to  $r$ ) and

$$t_3 = -\frac{w}{6} \quad (9.13)$$

Since  $m = \frac{w}{2}$

$$t_3 = -\frac{m}{3} \quad (9.14)$$

Finally in the knot representation the electric charge is by (9.1) and (9.8)

$$Q_a = -\frac{e}{3} (m + m') \quad (9.15)$$

But in the standard theory (point particle representation)

$$Q = (t_3 + t_0)e \quad (9.16)$$

Since (9.15) and (9.16) must agree, we have

$$t_3 + t_0 = -\frac{1}{3} (m + m') \quad (9.17)$$

By (9.14) and (9.17) the hypercharge is

$$t_0 = -\frac{1}{3} m' \quad (9.18)$$

Therefore alternative forms of the quantum state of the fermionic knots are

$$D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} \quad \text{or} \quad D_{-3t_3 - 3t_0}^{3t} \quad (9.19)$$

Therefore the invariance group of the algebra, namely  $U_a(1) \times U_b(1)$ , defines the charge and hypercharge.

The preceding relations may be summarized as follows:

$$\left\{ \begin{array}{l} t = \frac{N}{6} \\ t_3 = -\frac{w}{6} \\ t_0 = -\frac{r+1}{6} \\ Q_e = -\frac{e}{6}(w + r + 1) \end{array} \right. \quad (9.20) \quad \text{or} \quad \left\{ \begin{array}{l} t = \frac{j}{3} \\ t_3 = -\frac{m}{3} \\ t_0 = -\frac{m'}{3} \\ Q_e = -\frac{e}{3}(m + m') \end{array} \right. \quad (9.21)$$

For trefoils  $N = 3$  and  $j = \frac{N}{2} = \frac{3}{2}$ . The factor  $\frac{1}{3}$  appears because  $N = 3$  and independently because the quantum of charge is  $e/3$ . The additional factor of  $1/2$  in the  $1/6$  factor appears because we are describing fermions by  $1/2$  integer representations.

Note also that

$$Q_e = -\frac{e}{N} \left( \frac{w + r + 1}{2} \right) \quad \text{or} \quad -e \left( \frac{m + m'}{2j} \right) \quad \text{and} \quad D_{mm'}^j = D_{-3t_3 - 3t_0}^{3t} \quad (9.22)$$

hold for all the fermionic knots.

We may also define the “writhe charge” and the “rotation charge”:

$$Q(w) \equiv -k \frac{w}{2} \quad (9.23)$$

$$Q(r) \equiv -k \frac{r+1}{2} \quad (9.24)$$

Then we have

$$Q_a = Q(w) + Q(r) \quad (9.25)$$

$$Q_b = Q(w) - Q(r) \quad (9.26)$$

$$Q_e = Q(w) + Q(r) \quad (9.27)$$

and

$$Q(w) = et_3 \quad (9.28)$$

$$Q(r) = et_0 \quad (9.29)$$

i.e.  $t_3$  and  $t_0$  measure the “writhe charge” and “rotation charge” respectively and the electric charge measures the sum.

The earlier equations (6.6b) and (9.3) may be rewritten in terms of  $Q(w)$  and  $Q(r)$  as

$$D^{\frac{N}{2}}_{\frac{w}{2} \frac{r+1}{2}}(a'b'c'd') = e^{-\frac{i}{k}Q(w)\varphi_w} e^{-\frac{i}{k}Q(r)\varphi_r} D^{\frac{N}{2}}_{\frac{w}{2} \frac{r+1}{2}}(abcd) \quad (9.30)$$

Finally, by (8.18), (8.19) and (9.21) we define the charge operator

$$Q = -\frac{e}{3} \frac{\mathcal{W} + \mathcal{R}}{2} \left( = -\frac{e}{3}(m + m') \right) \quad (9.31)$$

and by (8.15) and (8.16)

$$= -\frac{e}{3}(\omega_a - \omega_d) \quad \text{with } SL_q(2) \quad (9.32)$$

$$= -\frac{e}{3}(\omega_a - \omega_{\bar{a}}) \quad \text{with } SU_q(2) \quad (9.33)$$

and by (9.19) and (9.31)

$$Q D^{3t}_{-3t_3-3t_0} = e(t_3 + t_0) D^{3t}_{-3t_3-3t_0} \quad (9.34)$$

## 10 The Fermion-Boson Interactions<sup>11,12</sup>

To discuss interactions we introduce a knot field theory by replacing the point particles of standard theory with quantum knots. This is done by attaching to each normal mode a knot state just as one introduces spin by attaching a spin state. The knot states and therefore the corresponding fields are represented by operators defined only up to the gauge transformations (9.4) and (9.5), and we shall require the action to be invariant under these gauge transformations, since they are induced by the underlying transformations that leave the defining algebra invariant. Therefore, by Noether’s theorem,  $Q(w)$  and  $Q(r)$  behave, in the field theory as well, as conserved charges, consistent with their identification as topological charges.

We therefore assume that the topological charges associated with the knot gauge group are conserved by the emission and absorption of bosonic solitons, which also carry topological charge, as a consequence of the following fermion-boson interaction.

$$\bar{\mathcal{F}}_3 \mathcal{B}_2 \mathcal{F}_1 \quad (10.1)$$

where

$$\mathcal{F}_1 = F_1(p, s, t) D_{m_1 m'_1}^{3/2}(abcd) |n_1\rangle \quad (10.2)$$

$$\bar{\mathcal{F}}_3 = \langle n_3 | \bar{D}_{m_3 m'_3}^{3/2}(abcd) \bar{F}_3(p, s, t) \quad (10.3)$$

$$\mathcal{B}_2 = B_2(p, s, t) D_{m_2 m'_2}^j(abcd) \quad (10.4)$$

Here  $F(p, s, t)$  and  $B(p, s, t)$  are the standard fermionic and bosonic normal modes where  $p$  and  $s$  refer to momentum and spin. Then (10.1) becomes

$$(\bar{F}_3 B_2 F_1) \langle n_3 | \bar{D}_{m_3 m'_3}^{3/2} D_{m_2 m'_2}^j D_{m_1 m'_1}^{3/2} | n_1 \rangle \quad (10.5)$$

The correction to the standard matrix elements appears in the second factor, namely

$$\langle n_3 | \bar{D}_{m_3 m'_3}^{3/2} D_{m_2 m'_2}^j D_{m_1 m'_1}^{3/2} | n_1 \rangle \quad (10.6)$$

If there are  $M$  generations of fermions, then  $n_1$  and  $n_3$  take on values  $0, \dots, M-1$ . All present evidence appears to favor  $M=3$ . We shall assume that the only occupied states  $|n\rangle$  are  $n=0, 1, 2$  in order of increasing mass.

We require that the basic internal interaction be invariant under gauge transformations,  $U_a(1) \times U_b(1)$ , of the underlying algebra, i.e.

$$\left( \bar{D}_{m_3 m'_3}^{3/2} \right)' \left( D_{m_2 m'_2}^j \right)' \left( D_{m_1 m'_1}^{3/2} \right)' = \bar{D}_{m_3 m'_3}^{3/2} D_{m_2 m'_2}^j D_{m_1 m'_1}^{3/2} \quad (10.7)$$

Then by (9.4) and (9.5)

$$\exp[ik\varphi_a](-Q_a(3) + Q_a(2) + Q_a(1)) = 1 \quad (10.8)$$

$$\exp[ik\varphi_b](-Q_b(3) + Q_b(2) + Q_b(1)) = 1 \quad (10.9)$$

Therefore both  $Q_a$  and  $Q_b$  are conserved:

$$Q(1) + Q(2) = Q(3) \quad (10.10)$$

Then by (9.1) and (9.2)

$$m_3 = m_1 + m_2 \quad (10.11)$$

$$m'_3 = m'_1 + m'_2 \quad (10.12)$$

and the possible values of  $(j, m, m')$  for the intermediate boson are restricted by the known values of  $(j, m, m')$  for the initial and final fermions.

If the rules for connecting  $m$  and  $m'$  to  $t_3$  and  $t_0$  are extended without change from fermions to the intermediate boson, then the conservation of  $Q_a$  and  $Q_b$  by the basic interaction implies the conservation of  $t_3$  and  $t_0$  by the same interaction. Therefore we adopt for bosonic knots the same rules as for fermionic knots:

$$\begin{aligned} m &= -3t_3 \\ m' &= -3t_0 \\ j &= 3t \end{aligned} \tag{10.13}$$

Applied to the vector bosons these rules imply Table 10.1. The first three columns of Table 3 express the fact that  $\vec{W}$  is an isotriplet and  $W^0$  is an isosinglet in the standard theory.

The fourth column  $D_{-3t_3 -3t_0}^{3t}$  labels the internal states of the four vector bosons. If  $t = 1, j = 3$ ; and if  $j = \frac{N}{2}$ , as we have assumed, then  $N = 6$  and  $W$  is a ditrefoil consistent with the pair production of fermions by vector bosons.

**Table 10.1**

	$t$	$t_3$	$t_0$	$D_{-3t_3 -3t_0}^{3t}$
$W^+$	1	1	0	$D_{-3 \ 0}^3$
$W^-$	1	-1	0	$D_{3 \ 0}^3$
$W^3$	1	0	0	$D_{0 \ 0}^3$
$W^0$	0	0	0	$D_{0 \ 0}^0$

Note

$$Q_e = -\frac{e}{N}(w + r + 1) \tag{10.14}$$

for the vector bosons, corresponding to (9.22) for fermions.

Since  $W^0$  is a  $U(1)$  coupling in the standard theory, there is no self-coupling, i.e. it itself carries neither electric nor hypercharge. The assignment of  $j$  to  $W^0$  is also restricted in the internal matrix element by the  $q$ -Clebsch-Gordan rules.<sup>13,14</sup> If  $j = 0$  and we maintain the relation  $j = N/2$ , then  $N = 0$  and  $W^0$  is an unknotted clockwise loop.

The possibility of extending the conservation laws and the same rule for associating  $Q_a$  and  $Q_b$  with  $m$  and  $m'$  to all solitons, as here defined, depends on the fact that  $Q_a$  and  $Q_b$  depend solely on  $m$  and  $m'$  and are independent of  $j$ .

The conservation of  $Q_a$  and  $Q_b$ , or equivalently of  $Q_w$  and  $Q_r$ , i.e. the invariance of the action under  $U_a(1) \otimes U_b(1)$  is the origin in the knot model of the conservation of  $t_3$  and  $t_0$ . Electric charge and hypercharge in this model are characterizations of the topology of the knotted soliton. Since we are describing a modified standard model, it is essential that  $t_3$  and  $t_0$  defined by  $U_a(1) \times U_b(1)$  agree with the  $t_3$  and  $t_0$  defined by  $SU(2) \times U(1)$ . This agreement is expressed in (10.13).

## 11 Fermion and Boson State Functions<sup>11</sup>

According to (10.5) and (10.6) of the knot model the standard matrix elements are modified by the form factors (10.6). To compute these form factors one needs the state functions of the fermionic knots as well as the state functions of the bosonic knots. They are shown in Tables 11.1 and 11.2 computed according to (5.19) expressed as follows:

$$D_{mm'}^j = \Delta_{mm'}^j \Sigma_{mm'}^j \quad (11.1)$$

where

$$\Delta_{mm'}^j = \left[ \frac{\langle n'_+ \rangle_1 \langle n'_- \rangle_1}{\langle n_+ \rangle_1 \langle n_- \rangle_1} \right]^{1/2} \quad (11.2)$$

$$\Sigma_{mm'}^j = \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_1 \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_1 \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \quad (11.3)$$

Here

$$\left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_1 = \left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_{q_1}$$

where  $\left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_q$  is given by (5.3). In the following tables and applications we shall pass from  $SL_q(2)$  to  $SU_q(2)$  by setting

$$\begin{aligned} d &= \bar{a} \\ c &= -q_1 \bar{b} \end{aligned}$$

in (11.3). We record these form factors for a few of the elementary processes.

**Table 11.1**

*Fermions:*

	$\underline{(e\mu\tau)}$	$\underline{(dsb)}$	$\underline{(uct)}$	$\underline{\nu_e\nu_\mu\nu_\tau}$
$(w, r) :$	$(3, 2)$	$(3, -2)$	$(-3, -2)$	$(-3, 2)$
$D_{\frac{w}{2}\frac{r+1}{2}}^{N/2} :$	$D_{\frac{3}{2}\frac{3}{2}}^{3/2}$	$D_{\frac{3}{2}\frac{1}{2}}^{3/2}$	$D_{-\frac{3}{2}\frac{1}{2}}^{3/2}$	$D_{-\frac{3}{2}\frac{3}{2}}^{3/2}$
$D_{\frac{w}{2}\frac{r+1}{2}}^{N/2} :$	$a^3$	$\Delta_{-\frac{3}{2}\frac{1}{2}}^{3/2} \left\langle \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right\rangle_1 ab^2$	$-\Delta_{-\frac{3}{2}\frac{1}{2}}^{3/2} \left\langle \begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right\rangle_1 q_1 \bar{b}\bar{a}^2$	$-q_1^3 \bar{b}^3$

**Table 11.2**

*Bosons:*

	$\underline{W^-}$	$\underline{W^+}$	$\underline{W^3}$	$\underline{W^0}$
$D_{-3t_3-3t_0}^{3t} :$	$D_{30}^3$	$D_{-30}^3$	$D_{00}^3$	$D_{00}^0$
$D_{-3t_0-3t_0}^{3t} :$	$\Delta_{30}^3 \left\langle \begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \right\rangle_1 a^3 b^3$	$-\Delta_{-30}^3 \left\langle \begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \right\rangle_1 q_1^3 \bar{b}^3 \bar{a}^3$	$f_3(\bar{b}b)$	$f_0(\bar{b}b)$

where

$$\Delta_{-30}^3 = \Delta_{30}^3 = \frac{\langle 3 \rangle_1}{\langle 6 \rangle_1^{1/2}} \quad \text{and} \quad \left\langle \begin{smallmatrix} 6 \\ 3 \end{smallmatrix} \right\rangle_1 = \frac{\langle 6 \rangle_1!}{\langle 3 \rangle_1! \langle 3 \rangle_1!}$$

Here

$$\langle n \rangle_1 = 1 + q_1 + q_1^2 + \dots + q_1^{n-1}$$

In this table  $f_3$  and  $f_0$  are polynomials in  $\bar{b}b$  that may be computed by (11.1)-(11.3).

## 12 Lepton Neutrino Couplings<sup>11</sup>

Let us consider the process

$$\bar{\ell}(j) + W^- \rightarrow \bar{\nu}(i) \quad \text{or} \quad W^- \rightarrow \ell(j) + \bar{\nu}(i) \quad (12.1)$$

The standard matrix element for the absorption of a  $W^-$  boson by a  $\bar{\ell}(j)$  with the emission of a  $\bar{\nu}(i)$  is by (10.6) to be multiplied by the following factor

$$m(i, j) = \langle i | \bar{D}_{-\frac{3}{2}\frac{3}{2}}^{\frac{3}{2}} D_{30}^3 \bar{D}_{\frac{3}{2}\frac{3}{2}}^{\frac{3}{2}} | j \rangle \quad (12.2)$$



where the double bar signifies an antiparticle in the final state. By the Tables (11.1) and (11.2)

$$m(i, j) = \Delta_{30}^3 \left\langle \begin{matrix} 6 \\ 3 \end{matrix} \right\rangle_1 q_1^3 \langle i | (-\bar{b}^3)(a^3 b^3)(\bar{a}^3) | j \rangle \quad (12.3)$$

or

$$m(i, j) = m(n) \delta(i, j) \quad (12.4)$$

where

$$m(n) = - \left\langle \begin{matrix} 6 \\ 3 \end{matrix} \right\rangle_1^{1/2} q^{6+6n} |\beta|^6 f(n) f(n+1) f(n+2) \quad (12.5)$$

and

$$n = n_i = n_j \quad (12.6)$$

Here

$$f(n) = 1 - q^{2n} |\beta|^2 \quad (12.7)$$

The form factor  $m(i, j)$  vanishes if  $i \neq j$  and depends on  $n$  as shown. Then define

$$R_n \equiv \frac{m(n+1)}{m(n)} = q^6 \frac{1 - q^{2n+6} |\beta|^2}{1 - q^{2n} |\beta|^2} \quad (12.8)$$

including

$$R_0 = q^6 \frac{1 - q^6 |\beta|^2}{1 - |\beta|^2} \quad (12.9)$$

and

$$R_1 = q^6 \frac{1 - q^8 |\beta|^2}{1 - q^2 |\beta|^2} \quad (12.10)$$

or by (12.9)

$$|\beta|^2 = \frac{R_0 - q^6}{R_0 - q^{12}} \quad (12.11)$$

and by (12.10)

$$|\beta|^2 = q^{-2} \frac{R_1 - q^6}{R_1 - q^{12}} \quad (12.12)$$

The universal Fermi interaction requires

$$R_0 = R_1 = 1 \quad (12.13)$$

and implies by (12.11) and (12.12)

$$\begin{aligned} q &= 1 \\ |\beta| &= \frac{\sqrt{2}}{2} = .707 \end{aligned} \quad (12.14)$$

If the universal Fermi interaction is not exactly satisfied, then the values of  $(q, \beta)$  as determined by  $R_0$  and  $R_1$  will be shifted slightly away from (12.14).

### 13 Charge Changing Quark Couplings<sup>11</sup>

Let  $Q\left(-\frac{1}{3}, j\right)$  be any quark of charge  $-\frac{1}{3}e$  and let  $Q\left(\frac{2}{3}, j\right)$  be any quark of charge  $+\frac{2}{3}e$ . Then we consider

$$Q\left(-\frac{1}{3}, j\right) + W^+ \rightarrow Q\left(\frac{2}{3}, i\right) \quad (13.1)$$

and denote its form factor by

$$\langle i | \bar{D}_{-\frac{3}{2}-\frac{1}{2}}^{3/2} D_{-30}^3 D_{\frac{3}{2}-\frac{1}{2}}^{3/2} | j \rangle = m(n) \delta(n, n_i) \delta(n, n_j) \quad (13.2)$$

where

$$m(n) = -C q^{6n+4} |\beta|^6 f(n) f(n+1) f(n-1) \quad (13.3)$$

Here the quark states are denoted by

$$D_{\frac{3}{2}-\frac{1}{2}}^{3/2} | i \rangle \quad \text{and} \quad D_{-\frac{3}{2}-\frac{1}{2}}^{3/2} | i \rangle \quad (13.4)$$

and the corresponding state of  $W^+$  by

$$D_{-30}^3$$

In (13.3)

$$C = \Delta_{\frac{3}{2}-\frac{1}{2}}^{3/2} \Delta_{-\frac{3}{2}-\frac{1}{2}}^{3/2} \left( \left\langle \begin{matrix} 3 \\ 1 \end{matrix} \right\rangle_1 \right)^2 \cdot \Delta_{-30}^3 \left\langle \begin{matrix} 6 \\ 3 \end{matrix} \right\rangle_1 \quad (13.5)$$

and  $\Delta_{mm'}^i$  is given by (11.2). Again defining  $R_n$  by

$$R_n = \frac{m(n+1)}{m(n)} \quad (13.6)$$

we have

$$R_0 = q^6 \frac{1 - q^4 |\beta|^2}{1 - q^{-2} |\beta|^2} = \frac{m(s + W^+ \rightarrow c)}{m(d + W^+ \rightarrow u)} \quad (13.7)$$

$$R_1 = q^6 \frac{1 - q^6 |\beta|^2}{1 - |\beta|^2} = \frac{m(b + W^+ \rightarrow t)}{m(s + W^+ \rightarrow c)} \quad (13.8)$$

Then

$$|\beta|^2 = q^2 \frac{R_0 - q^6}{R_0 - q^{12}} \quad \text{and} \quad |\beta|^2 = \frac{R_1 - q^6}{R_1 - q^{12}} \quad (13.9)$$

We again have

$$\begin{aligned} q &= 1 \\ |\beta| &= \frac{1}{2}\sqrt{2} = .707 \end{aligned} \tag{13.10}$$

if  $R_0 = R_1 = 1$  but according to the Kobayashi-Maskawa matrix we have

$$R_0 = \left( \frac{.973}{.974} \right)^{1/2} = .999 \tag{13.11}$$

$$R_1 = \left( \frac{.999}{.973} \right) = 1.01 \tag{13.12}$$

Since the diagonal elements of the Kobayashi-Maskawa matrix are not quite equal,  $q$  and  $|\beta|$  differ slightly from  $q = 1, |\beta| = .707$ , if these Kobayashi-Maskawa ratios are attributed entirely to the knot form factors.

**Table 13.1: The Kobayashi-Maskawa (KM) Matrix**

	$d$	$s$	$b$
$u$	0.974	0.226	0.00359
$c$	0.226	0.973	0.0415
$t$	0.009	0.0407	0.999

The elements (13.2) are taken between states  $|b'\rangle$  that are eigenstates of the Hamiltonian  $H(\bar{b}, b)$  and therefore may be regarded as states of definite mass. One may similarly regard the eigenstates of  $d$  and  $a$ , that are raising and lowering, or creation and annihilation operators, as flavor states, which are superpositions of mass states. We have seen that matrix elements like (13.2), describing transitions between quarks that are mediated by weak bosons, are diagonal when taken between the mass eigenstates  $|b'\rangle$ . They are not diagonal, however, in accord with Table (13.1), when taken between flavor states as follows:

$$\langle d'|M|a'\rangle = \sum_{b'b''} \langle d'|b'\rangle \langle b'|M|b''\rangle \langle b''|a'\rangle$$

If one requires that  $\langle d'|M|a'\rangle$ , taken between flavor states, be a  $SU(3)$  matrix, then it is natural to parametrize by  $q, |\beta|$ , and the three complex eigenvalues of  $a$ .<sup>19</sup>

## 14 The Preon Representation<sup>15,16</sup>

The model so far described is based on representations  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$  of the knot algebra  $SL_q(2)$  or  $SU_q(2)$ , where the integers  $(N, w, r)$  label classical knots. With these representations we define  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} |n\rangle$  to be the state of a “quantum knot” where  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}(a, b, c, d)$  may be regarded as a kinematic factor. This state is additionally an eigenstate of the Hamiltonian that defines the dynamics of the knot.

Since the empirical basis of these considerations is the correspondence between the 4 quantum trefoils and the 4 families of fermions, we have focused on trefoils ( $N = 3$ ) and therefore on  $j = \frac{N}{2} = \frac{3}{2}$ . As we have seen the 4 families (neutrinos, leptons, up quarks, down quarks) may all be represented by elements of the  $3/2$  representation of  $SL_q(2)$ . They can also all be represented by  $D_{-3t_3-3t_0}^{3t}$  corresponding to the fact that the isotopic spin  $t = \frac{1}{2}$  for all the elementary fermions.

It is then natural to examine the adjoint ( $j = 1$ ) and fundamental ( $j = 1/2$ ) representations. To do this we extend the results found for the  $3/2$  representation, i.e. we set

$$j = \frac{N}{2} = 3t \quad (14.1)$$

$$m = \frac{w}{2} = -3t_3 \quad (14.2)$$

$$m' = \frac{r+1}{2} = -3t_0 \quad (14.3)$$

$$Q = -\frac{e}{3}(m + m') = -\frac{e}{3}(w + r + 1) = e(t_3 + t_0) \quad (14.4)$$

According to Eqs. (11.1)-(11.3) and ignoring numerical factors, the fundamental and adjoint representations are shown in Tables (14.1) and (14.2).

**Table 14.1**

$m \backslash m'$	$\frac{1}{2}$	$-\frac{1}{2}$
$\frac{1}{2}$	$a$	$b$
$-\frac{1}{2}$	$c$	$d$

$D_{mm'}^{1/2} :$

**Table 14.2**

$m \backslash m'$	1	0	-1
1	$a^2$	$ab$	$b^2$
0	$ac$	$ad + bc$	$bd$
-1	$c^2$	$cd$	$d^2$

$D_{mm'}^1 :$

According to (14.2)-(14.4) one finds the values of  $(t_3, t_0, Q)$  shown in Tables (14.3) and (14.4)

**Table 14.3**

	$\underline{t_3}$	$\underline{t_0}$	$\underline{Q}$
$a$	$-\frac{1}{6}$	$-\frac{1}{6}$	$-\frac{\epsilon}{3}$
$c$	$\frac{1}{6}$	$-\frac{1}{6}$	$0$
$d$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{\epsilon}{3}$
$b$	$-\frac{1}{6}$	$\frac{1}{6}$	$0$

**Table 14.4**

	$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$		$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$		$\underline{t_3}$	$\underline{t_0}$	$\underline{Q/e}$	$\underline{D_{mm'}^1}$
$D_{11}^1$	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{2}{3}$	$a^2$	$D_{01}^1$	$0$	$-\frac{1}{3}$	$-\frac{1}{3}$	$ac$	$D_{-11}^1$	$\frac{1}{3}$	$-\frac{1}{3}$	$0$	$c^2$
$D_{10}^1$	$-\frac{1}{3}$	$0$	$-\frac{1}{3}$	$ab$	$D_{00}^1$	$0$	$0$	$0$	$ad + bc$	$D_{-10}^1$	$\frac{1}{3}$	$0$	$\frac{1}{3}$	$cd$
$D_{1-1}^1$	$-\frac{1}{3}$	$\frac{1}{3}$	$0$	$b^2$	$D_{0-1}^1$	$0$	$\frac{1}{3}$	$\frac{1}{3}$	$bd$	$D_{-1-1}^1$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$d^2$

In these tables  $a$  and  $d$  have opposite values of the charge and hypercharge, while  $c$  and  $b$  are both neutral with opposite values of  $t_3$  and  $t_0$ .

We next relate the fundamental and adjoint representations to the knot representations shown in Table 14.5.

**Table 14.5**

Elementary Fermions	$D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$	$D_{\frac{w}{2} \frac{r+1}{2}}^{3/2}$	$\underline{Q_e}$	$\underline{et_0}$
$(e^-, \mu^-, \tau^-)$	$D_{\frac{3}{2} \frac{3}{2}}^{3/2}$	$a^3$	$-e$	$-\frac{\epsilon}{2}$
$(\nu_e, \nu_\mu, \nu_\tau)$	$D_{-\frac{3}{2} \frac{3}{2}}^{3/2}$	$c^3$	$0$	$-\frac{\epsilon}{2}$
$(d, s, b)$	$D_{\frac{3}{2} -\frac{1}{2}}^{3/2}$	$\sim ab^2$	$-\frac{1}{3}e$	$\frac{\epsilon}{6}$
$(u, c, t)$	$D_{-\frac{3}{2} -\frac{1}{2}}^{3/2}$	$\sim cd^2$	$\frac{2}{3}e$	$\frac{\epsilon}{6}$

In Table 14.5 the four monomials in the knot algebra represent the four fermionic knots. This table may also be interpreted by regarding the element  $a$  as a creation operator for

a preon of charge  $-e/3$ , and hypercharge  $-e/6$  and by regarding  $d$  as a creation operator for a preon of charge  $+e/3$  and hypercharge  $+e/6$  while  $b$  and  $c$  are regarded as creation operators for neutral preons with hypercharge  $e/6$  and  $-e/6$  respectively. This interpretation is consistent with the charge and hypercharge assignments in Tables 14.3 and 14.4 and also with our conclusion from earlier work that adjoint operators  $(a, d)$  correspond to opposite charges and that the  $(b, c)$  sector describes neutral states. According to the same picture the fermion knots, like the nucleons, are composed of three fermions, which are now preons.

The results shown in these tables illustrate the following general statements that can be proved for any knot described by  $D_{-3t_3-3t_0}^{3t}(a, b, c, d)$ :

$$t_3 = -\frac{1}{6}(n_a - n_d + n_b - n_c) \quad (14.5)$$

$$t_0 = -\frac{1}{6}(n_a - n_d - n_b + n_c) \quad (14.6)$$

$$Q = \frac{e}{3}(n_d - n_a) \quad (14.7)$$

where  $(n_a, n_b, n_c, n_d)$  are the exponents of  $(a, b, c, d)$  respectively in (5.19). Since the  $(a, b, c, d)$  are now interpreted as creation operators for  $(a, b, c, d)$  preons, the  $(n_a, n_b, n_c, n_d)$  are now the numbers of  $(a, b, c, d)$  particles. The  $(n_a, n_b, n_c, n_d)$  vary among the terms contributing to  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$  but  $n_a - n_d$ ,  $n_b - n_c$ , and  $n_a + n_b + n_c + n_d$  are the same for every term, and therefore characterize  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} = D_{-3t_3-3t_0}^{3t}$ .

In addition to (14.5)-(14.7) one has

$$\begin{aligned} N' &\equiv n_a + n_b + n_c + n_d \\ &= 2j = N \end{aligned} \quad (14.8)$$

where  $N'$  is the total number of preons in the knot. **Therefore the total number of preons ( $N'$ ) in the knot described by  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$  is equal to the number of crossings ( $N$ ).** Since the preons are to be regarded as fermions,  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$  represents a boson or a fermion depending on whether the number of crossings is even or odd, as we have previously assumed.

The relations (14.5)-(14.7) may be shown as follows. By (8.17)-(8.19) the operators  $(\mathcal{N}, \mathcal{W}, \mathcal{R})$  have eigenvalues  $(2j, 2m, 2m')$  and if one extends (9.19)-(9.21) to all representations one has

$$(2j, 2m, 2m') = (6t, -6t_3, -6t_0) = (N, w, r + 1) \quad (14.9)$$

On the other hand, the eigenvalues of  $(\mathcal{N}, \mathcal{W}, \mathcal{R})$  when these operators are defined by (8.13)-(8.16), may also be expressed as

$$(n_a + n_b + n_c + n_d; n_a - n_d + n_b - n_c; n_a - n_d - n_b + n_c) \quad (14.10)$$

Then (14.9) and (14.10) imply (14.5)-(14.7) and (14.8).

If we maintain the relation  $j = N/2$ , then  $j = 1/2$  and  $j = 1$  imply values of  $N < 3$ . Since the minimum value of  $N$  is three for a classical knot,  $j = 1/2$  and  $j = 1$  do not qualify as images of classical knots. They may be pictured as quantum images of twisted loops. Viewed as a particle a fermion becomes a boson by emitting or absorbing a preon. Viewed as a fermionic knot it becomes a bosonic knot by adding or subtracting a curl. A curl in turn is a twisted loop that has been cut.

Although our definition of the quantum knot is based on the knot algebra, it does not follow that the quantum knot closely resembles the geometrical knot, just as the quantum harmonic oscillator does not closely resemble the classical harmonic oscillator. In particular, although the fundamental  $D^{1/2}$  and adjoint  $D^1$  representations have  $N < 3$  and therefore do not qualify as images of knots in the classical sense, they are not thereby disqualified as physical just as zero-point oscillations of the harmonic oscillator are not physically disqualified. Therefore we shall take the view that the  $D_{mm'}^{1/2}$  fermions and the  $D_{mm'}^1$  bosons qualify as particles of new fields and we shall assume that they are subject to the Lagrangian of the standard theory and may be discussed in the same way as the fermions and bosons of standard theory.

## 15 Preons as Physical Particles<sup>15</sup>

We shall now no longer regard the preons as merely a simple way to describe the algebraic structure of the knot polynomials. If these preons are in fact physical particles, the following decay modes of the quarks are possible

$$\begin{array}{lll} \text{down quarks : } D_{\frac{3}{2}-\frac{1}{2}}^{3/2} \longrightarrow D_{\frac{1}{2}\frac{1}{2}}^{1/2} + D_{1-1}^1 & ab^2 \longrightarrow a + b^2 \\ & \text{or} \\ \text{up quarks : } D_{-\frac{3}{2}-\frac{1}{2}}^{3/2} \longrightarrow D_{-\frac{1}{2}\frac{1}{2}}^{1/2} + D_{-1-1}^1 & cd^2 \longrightarrow c + d^2 \end{array}$$

and the preons could play an intermediary role as virtual particles in quark processes.

The justification for considering the preons seriously as physical particles would then no longer depend on the knot conjecture but rather on a more general role of  $SL_q(2)$  gauge invariance. Then the preons would appear as matrix elements of the fundamental and adjoint representations of  $SL_q(2)$  just as the fermionic and bosonic quantum knots appear in the  $j = 3/2$  and  $j = 3$  representations of  $SL_q(2)$ .<sup>3</sup> In this scenario quantum knots would be just one of the manifestations of a  $SL_q(2)$  related symmetry. There would also be no need to introduce a new Lagrangian for the preons since all particles described by representations of  $SL_q(2)$  would be subject to the same modified standard action.

The simple knot model predicts an unlimited number of excited states<sup>2,3</sup> but it appears that there are only three generations, e.g.  $(d, s, b)$ . According to the preon scenario, however, it may be possible to avoid this problem by showing that the quarks will dissociate into preons if given a critical “dissociation energy” less than that needed to reach the level of the fourth predicted flavor. In that case one would also expect the formation of a preon-quark plasma at sufficiently high temperatures. It may be possible to study the thermodynamics of the plasma composed of quarks and these hypothetical particles.

Since the  $a$  and  $\bar{a}$  particles are charged ( $\pm e/3$ ) one should also expect their electroproduction according to

$$e^+ + e^- \rightarrow a + \bar{a} + \dots$$

at sufficiently high energies of a colliding  $(e^+, e^-)$  pair. More generally one should expect

$$? \rightarrow \gamma \rightarrow a + \bar{a} + \dots$$

if  $\gamma$  is sufficiently energetic independent of how it is produced.

## 16 Field Theory of Quantum Knots<sup>16</sup>

Let us introduce the field  $\Psi_{mm'}^j(x; abcd)$ , the product of the standard point particle field,  $\psi_{mm'}^j(x)$ , and an internal factor  $D_{mm'}^j$ , as follows:

$$\Psi_{mm'}^j = \psi_{mm'}^j(x) D_{mm'}^j(a, b, c, d) \tag{16.1}$$



These fields undergo transformations of the Poincaré algebra when the spacetime points  $(x)$  are relabelled by transformations that preserve the structure of spacetime, and they also undergo gauge transformations when the discrete elements  $(a, b, c, d)$  are transformed so as to preserve the structure of the knot algebra. By the usual argument the Lagrangian must be constructed to be invariant under all of these transformations since the relabelling of the continuum and the algebra must not influence the physics. There is then, by Noether's theorem, a conserved quantity for each independent gauge transformation. In the familiar way the eigenvalues of the corresponding conserved and commuting operators are used to label the particles and some of these eigenvalues are functions of the  $(j, m, m')$  that label the knot particles.

In much of the following discussion we do not distinguish between  $SL_q(2)$  and  $SU_q(2)$  but in part of the work we may explicitly refer to  $SU_q(2)$  by setting  $d = \bar{a}$  and  $c = -q_1 \bar{b}$ .

In the knot electroweak theory, just as in the standard model, the fields representing the fermion families may be arranged by (16.1) in two isotopic doublets as follows:

$$\begin{aligned} \text{leptons : } \Psi_\ell &= \psi_\ell(x) D_{\frac{3}{2} \frac{3}{2}}^{3/2}(abcd) \\ \text{neutrinos : } \Psi_\nu &= \psi_\nu(x) D_{-\frac{3}{2} \frac{3}{2}}^{3/2}(abcd) \end{aligned} \quad (16.2)$$

and

$$\begin{aligned} \text{down quarks : } \Psi_d &= \psi_d(x) D_{\frac{3}{2} \frac{1}{2}}^{3/2}(abcd) \\ \text{up quarks : } \Psi_u &= \psi_u(x) D_{-\frac{3}{2} \frac{1}{2}}^{3/2}(abcd) \end{aligned} \quad (16.3)$$

Since the gauge transformations on the knot algebra induce corresponding gauge transformations on the  $D_{mm'}^j$  according to (6.6b), we then have in the lepton family

$$\Psi'_\ell = \psi_\ell(x) \left[ e^{i\frac{3}{2}(\varphi_a + \varphi_b)} e^{i\frac{3}{2}(\varphi_a - \varphi_b)} D_{\frac{3}{2} \frac{3}{2}}^{3/2}(abcd) \right] \quad (16.4)$$

or

$$\Psi'_\ell = e^{i\frac{3}{2}(\varphi_a + \varphi_b)} e^{i\frac{3}{2}(\varphi_a - \varphi_b)} \Psi_\ell \quad (16.5)$$

Likewise for the neutrino family we have

$$\Psi'_\nu = e^{-i\frac{3}{2}(\varphi_a + \varphi_b)} e^{i\frac{3}{2}(\varphi_a - \varphi_b)} \Psi_\nu \quad (16.6)$$

For the quarks we have

$$\Psi'_d = e^{i\frac{3}{2}(\varphi_a + \varphi_b)} e^{i\frac{1}{2}(\varphi_a - \varphi_b)} \Psi_d \quad (16.7)$$

$$\Psi'_u = e^{-i\frac{3}{2}(\varphi_a+\varphi_b)} e^{i\frac{1}{2}(\varphi_a-\varphi_b)} \Psi_u \quad (16.8)$$

The gauge transformations on the knot algebra therefore induce the following diagonal  $SU(2)$  transformations on the  $(\ell, \nu)$  and  $(u, d)$  doublets

$$\begin{aligned} \Psi'_\ell &= e^{i\frac{1}{2}\varphi_+} \Psi_\ell & \Psi'_d &= e^{i\frac{1}{2}\varphi_+} \Psi_d \\ \Psi'_\nu &= e^{-i\frac{1}{2}\varphi_+} \Psi_\nu & \Psi'_u &= e^{-i\frac{1}{2}\varphi_+} \Psi_u \end{aligned} \quad (16.9)$$

as well as the following  $U(1)$  transformations

$$\begin{aligned} \Psi'_\ell &= e^{i\frac{1}{2}\varphi_-} \Psi_\ell & \Psi'_d &= e^{i\frac{1}{6}\varphi_-} \Psi_d \\ \Psi'_\nu &= e^{i\frac{1}{2}\varphi_-} \Psi_\nu & \Psi'_u &= e^{i\frac{1}{6}\varphi_-} \Psi_u \end{aligned} \quad (16.10)$$

Here

$$\begin{aligned} \varphi_+ &= 3(\varphi_a + \varphi_b) \\ \varphi_- &= 3(\varphi_a - \varphi_b) \end{aligned} \quad (16.11)$$

In summary, the gauge transformations (G) on the algebra induce diagonal  $SU(2) \times U(1)$  transformations on the fermion doublets as follows:

$$\Psi'^{3t}_{-3t_3-3t_0} = e^{-it_3\varphi_+} e^{-it_0\varphi_-} \Psi^{3t}_{-3t_3-3t_0} \quad (16.12)$$

where

$$t = \frac{1}{2}, \quad t_3 = \pm \frac{1}{2}, \quad t_0 = \frac{1}{2}, \frac{1}{6} \quad (16.13)$$

The same Eq. (16.12) holds for the vector boson triplet with  $t = 1$  and the pair  $(t_3, t_0)$  as recorded in Table 10.1.

The gauge transformations (16.12) referring to the  $SL_q(2)$  or  $SU_q(2)$  are of course additional to the standard gauge transformations referring to isotopic  $SU(2)$ .

The statement (16.12) remains true if the transformations are local, i.e., if

$$\Psi'^{3t}_{-3t_3-3t_0} = e^{-it_3\varphi_+(x)} e^{-it_0\varphi_-(x)} \Psi^{3t}_{-3t_3-3t_0} \quad (16.14)$$

The preceding equation (16.14) permits the construction of an Abelian field theory based solely on the knot algebra.

The gauge transformations  $U_a \times U_b$  on the knot algebra may also be written in doublet form as follows:

$$\begin{pmatrix} a \\ c \end{pmatrix}' = \begin{pmatrix} e^{\frac{1}{6}\varphi_+} & 0 \\ 0 & e^{-\frac{1}{6}\varphi_+} \end{pmatrix} \begin{pmatrix} e^{\frac{1}{6}\varphi_-} & 0 \\ 0 & e^{\frac{1}{6}\varphi_-} \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} \quad (16.15a)$$

$$\begin{pmatrix} b \\ d \end{pmatrix}' = \begin{pmatrix} e^{\frac{i}{6}\varphi_+} & 0 \\ 0 & e^{-\frac{i}{6}\varphi_+} \end{pmatrix} \begin{pmatrix} e^{-\frac{i}{6}\varphi_-} & 0 \\ 0 & e^{-\frac{i}{6}\varphi_-} \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} \quad (16.15b)$$

where the components of the two doublets are preons.

Here

$$\varphi_a = \frac{1}{6}(\varphi_+ + \varphi_-) \quad (16.16a)$$

$$\varphi_b = \frac{1}{6}(\varphi_+ - \varphi_-) \quad (16.16b)$$

Transformations that mix  $a$  and  $c$  or  $b$  and  $d$ , however, do not leave the knot algebra (A) invariant. Hence (16.14) cannot be extended to off-diagonal  $SU(2)$  transformations, i.e. to

$$\Psi_{-3t_3-3t_0}^{3t}{}' = e^{-it\vec{\varphi}_+(x)} e^{-it_0\varphi_-(x)} \Psi_{-3t_3-3t_0}^{3t} \quad (16.17)$$

and therefore a non-Abelian field theory cannot be supported solely by the gauge group of the knot algebra. On the other hand, the isotopic spin  $\times$  hypercharge group,  $SU(2) \times U(1)$ , is empirically required and the standard electroweak theory postulates that this group is local.

If the vector field is introduced as the connection of this local group in the standard way, then in the  $q$ -knot modification of the standard theory, one may represent the vector fields by (16.1) where  $\psi_{mm'}^j$  transforms under local  $SU(2) \times U(1)$  and where the second factor in (6.1),  $D_{mm'}^j$ , transforms according to the global gauge symmetry of the knot algebra. The two symmetries, local  $SU(2) \times U(1)$  and global  $U(a) \times U(b)$ , are matched by requiring  $(j, m, m') = 3(t_1 - t_3, -t_0)$ .

## 17 Vector Fields and their Field Strengths

In the standard theory the vector bosons are quanta of the vector fields and the vector fields are connections of an underlying gauge group. In the standard electroweak model there are

two gauge groups, namely  $SU(2)$ , the isotopic spin group, and  $U(1)$ , the hypercharge group, and the corresponding vector connection of  $SU(2) \times U(1)$  is

$$W_+t_+ + W_-t_- + W_3t_3 + W_0t_0 \quad (17.1)$$

where the  $t_k$  are the generators of the Lie algebras of  $SU(2)$  and  $U(1)$ :

$$t_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad t_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad t_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (17.2)$$

By Eqs. (16.1) and (17.1) the corresponding connection in the knot model is

$$(W_+t_+)\mathcal{D}_+ + (W_-t_-)\mathcal{D}_- + (W_3t_3)\mathcal{D}_3 + (W_0t_0)\mathcal{D}_0 \quad (17.3)$$

$$= W_+(t_+\mathcal{D}_+) + W_-(t_-\mathcal{D}_-) + W_3(t_3\mathcal{D}_3) + W_0(t_0\mathcal{D}_0) \quad (17.4)$$

where the knot factors are

$$(\mathcal{D}_+, \mathcal{D}_-, \mathcal{D}_3, \mathcal{D}_0) \sim (D_{-30}^3, D_{30}^3, D_{00}^3, D_{00}^0)$$

and the  $D_{mm'}^j$  are given in Table 10.1. We are now referring explicitly to  $SU_q(2)$ .

Let us therefore define

$$\mathcal{W}_\mu = ig\vec{W}_\mu\vec{\tau} + ig_0W_\mu^0\tau_0 \quad (17.5)$$

where

$$\tau_\pm \equiv c_\pm t_\pm \mathcal{D}_\pm \quad (17.6)$$

$$\tau_3 \equiv c_3 t_3 \mathcal{D}_3 \quad (17.7)$$

$$\tau_0 \equiv c_0 t_0 \mathcal{D}_0 \quad (17.8)$$

and

$$\mathcal{D}_+ = \bar{b}^3 \bar{a}^3 \quad (17.9)$$

$$\mathcal{D}_- = a^3 b^3 \quad (17.10)$$

$$\mathcal{D}_3 = f(b\bar{b}) \quad (17.11)$$

$$\mathcal{D}_0 = 1 \quad (17.12)$$

Here the  $\mathcal{D}_k$  ( $k = +, -, 3$ ) differ from the  $D_{-3t_3-3t_0}^{3t}$  only by factors absorbed in the  $c_k$ .

The  $c_k$  will be determined in Section 19.

Let us define by (17.5) a covariant derivative

$$\nabla_\mu \equiv \partial_\mu + \mathcal{W}_\mu \quad (17.13)$$

that satisfies

$$\nabla'_\mu = S \nabla_\mu S^{-1} \quad (17.14)$$

where  $S \in SU(2) \times U(1) \times U_a(1) \times U_b(1)$ . Then

$$\mathcal{W}'_\mu = S \mathcal{W}_\mu S^{-1} + S \partial_\mu S^{-1} \quad (17.15)$$

The corresponding field strengths are

$$\mathcal{W}_{\mu\lambda} = (\nabla_\mu, \nabla_\lambda) \quad (17.16)$$

that transform as

$$\mathcal{W}'_{\mu\lambda} = S \mathcal{W}_{\mu\lambda} S^{-1} \quad (17.17)$$

We shall next ignore the  $W_\mu^0$  vector field. By (17.5), (17.13), and (17.16) the non-Abelian field strengths are

$$\mathcal{W}_{\mu\lambda} = ig(\partial_\mu W_\lambda^m - \partial_\lambda W_\mu^m)\tau_m - g^2 W_\mu^m W_\lambda^\ell [\tau_m, \tau_\ell] \quad (17.18)$$

Here

$$[\tau_k, \tau_\ell] = c_k c_\ell [[t_k, t_\ell] \mathcal{D}_k \mathcal{D}_\ell + t_\ell t_k [\mathcal{D}_k, \mathcal{D}_\ell]] \quad (17.19)$$

where

$$[t_k, t_\ell] = c_{k\ell}^s t_s \quad (k, \ell) = (+, -, 3) \quad (17.20)$$

$$[\mathcal{D}_k, \mathcal{D}_\ell] = \hat{c}_{k\ell}^s \mathcal{D}_s \quad (17.21)$$

$$t_k t_\ell = \gamma_{k\ell}^s t_s + \frac{1}{2} \delta(k, \pm) \delta(\ell, \mp) \quad (17.22)$$

$$\mathcal{D}_k \mathcal{D}_\ell = \hat{\gamma}_{k\ell}^s \mathcal{D}_s \quad (17.23)$$

Then

$$[\tau_k, \tau_\ell] = \frac{c_k c_\ell}{c_s} C_{k\ell}^s \tau_s + \frac{1}{2} c_k c_\ell \delta(k, \pm) \delta(\ell, \mp) \hat{c}_{k\ell}^s \mathcal{D}_s \quad (17.24)$$

where

$$C_{k\ell}^s = c_{k\ell}^s \hat{\gamma}_{k\ell}^s + \gamma_{\ell k}^s \hat{c}_{k\ell}^s \quad (17.25)$$

The structure coefficients of these algebras, including  $c_{k\ell}^s$  and  $\gamma_{\ell k}^s$  as well as the  $\hat{c}_{k\ell}^s$  and  $\hat{\gamma}_{k\ell}^s$ , commute since they are either numerically valued or are functions of  $\bar{b}b$ . They are all numerically valued when allowed to operate on states  $|n\rangle$  of the  $q$ -oscillator.

It follows from (17.18) and (17.24) that the field strengths are given by

$$\mathcal{W}_{\mu\lambda} = W_{\mu\lambda}^s \tau_s + \hat{W}_{\mu\lambda}^s \mathcal{D}_s \quad (17.26)$$

where

$$W_{\mu\lambda}^s = ig(\partial_\mu W_\lambda^s - \partial_\lambda W_\mu^s) - g^2 c_m c_\ell c_s^{-1} C_{m\ell}^s W_\mu^m W_\lambda^\ell \quad (17.27)$$

and

$$\hat{W}_{\mu\lambda}^s = -\frac{1}{2} g^2 c_m c_\ell \delta(\ell, \pm) \delta(m, \mp) \hat{c}_{m\ell}^s W_\mu^m W_\lambda^\ell \quad (17.28)$$

## 18 Interactions of the Vector Fields

(a) Self-Interactions.

We choose as the vector field invariant the following expectation value:

$$I = \langle 0 | \text{Tr } \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} | 0 \rangle \quad (18.1)$$

where  $|0\rangle$  is the ground state of the  $q$ -oscillator defined in Section 7. Here the trace is taken over the part dependent on the  $t_k$ .

To reduce I consider

$$\begin{aligned} \langle 0 | \mathcal{W}_{\mu\lambda} \mathcal{W}^{\mu\lambda} | 0 \rangle &= \sum_n \{ \langle 0 | W_{\mu\lambda}^s W^{r\mu\lambda} | n \rangle \langle n | \tau_s \tau_r | 0 \rangle \\ &\quad + \langle 0 | \hat{W}_{\mu\lambda}^s \hat{W}^{r\mu\lambda} | n \rangle \langle n | \mathcal{D}_s \mathcal{D}_r | 0 \rangle \\ &\quad + \langle 0 | W_{\mu\lambda}^s \hat{W}^{r\mu\lambda} | n \rangle \langle n | \tau_s \mathcal{D}_r | 0 \rangle \\ &\quad + \langle 0 | \hat{W}_{\mu\lambda}^s W^{r\mu\lambda} | n \rangle \langle n | \mathcal{D}_r \tau_s | 0 \rangle \} \end{aligned} \quad (18.2)$$

where the sum is over all states of the  $q$ -oscillator.

Since  $W_{\mu\lambda}^s$  and  $\hat{W}_{\mu\lambda}^s$  depend on the algebra of  $SU_q(2)$  only through  $\bar{b}b$ , they have no off-diagonal elements in  $n$ . Then

$$\begin{aligned} \text{Tr}\langle 0|\mathcal{W}_{\mu\lambda}\mathcal{W}^{\mu\lambda}|0\rangle &= \text{Tr}\{\langle 0|W_{\mu\lambda}^s W^{r\mu\lambda}|0\rangle\langle 0|\tau_s\tau_r|0\rangle \\ &+ \langle 0|\hat{W}_{\mu\lambda}^s \hat{W}^{r\mu\lambda}|0\rangle\langle 0|\mathcal{D}_s\mathcal{D}_r|0\rangle + \langle 0|W_{\mu\lambda}^s \hat{W}^{r\mu\lambda}|0\rangle \\ &\times \langle 0|\tau_s\mathcal{D}_r|0\rangle + \langle 0|\hat{W}_{\mu\lambda}^s W^{s\mu\lambda}|0\rangle\langle 0|\mathcal{D}_r\tau_s|0\rangle\} \end{aligned} \quad (18.3)$$

To continue the reduction of (18.1), we next compute the following factors in (18.3)

$$\langle 0|\text{Tr } \tau_s\tau_r|0\rangle = c_s c_r (\text{Tr } t_s t_r) \langle 0|\mathcal{D}_s\mathcal{D}_r|0\rangle \quad (18.4)$$

where

$$\langle 0|\mathcal{D}_s\mathcal{D}_r|0\rangle = [\delta(s, \pm)\delta(r, \mp) + \delta(s, 3)\delta(r, 3)]\langle 0|\bar{\mathcal{D}}_r\mathcal{D}_r|0\rangle \quad (18.5)$$

$$\langle 0|\text{Tr } \tau_s\mathcal{D}_r|0\rangle = \langle 0|\text{Tr } \mathcal{D}_r\tau_s|0\rangle = 0 \quad (18.6)$$

Then the field invariant reduces to

$$I = \sum_{s,r=(+,-)} \langle 0|A_{sr}W_{\mu\lambda}^s W^{r\mu\lambda} + 2\hat{W}_{\mu\lambda}^s \hat{W}^{r\mu\lambda}|0\rangle\langle 0|\mathcal{D}_s\mathcal{D}_r|0\rangle \quad (18.7)$$

where

$$A_{sr} = c_s c_r \text{Tr } t_s t_r \quad (18.8)$$

Here  $W_{\mu\lambda}^s$  and  $\hat{W}_{\mu\lambda}^s$  are given by (17.27) and (17.28) and the matrix elements  $\langle 0|\mathcal{D}_s\mathcal{D}_r|0\rangle = 0$  unless  $\mathcal{D}_s = \bar{\mathcal{D}}_r$ . Then

$$\langle 0|\bar{\mathcal{D}}_+\mathcal{D}_+|0\rangle = \langle 0|a^3 b^3 \bar{b}^3 a^3|0\rangle = q^{18}\langle 0|(\bar{b}b)^3 a^3 \bar{a}^3|0\rangle \quad (18.9)$$

$$\langle 0|\bar{\mathcal{D}}_-\mathcal{D}_-|0\rangle = \langle 0|\bar{b}^3 \bar{a}^3 a^3 b^3|0\rangle = \langle 0|\bar{b}b)^3 \bar{a}^3 a^3|0\rangle \quad (18.10)$$

$$\langle 0|\bar{\mathcal{D}}_3\mathcal{D}_3|0\rangle = |f(\bar{b}b)|^2 \quad (18.11)$$

where  $f(\bar{b}b)$  is given by (17.11) and abbreviates  $D_{00}^3$ . These matrix elements are all functions of  $\bar{b}b$  since

$$\bar{a}^n a^n = \prod_{s=1}^n (1 - q_1^{2s} \bar{b}b) \quad (18.12)$$

$$a^n \bar{a}^n = \prod_{s=0}^{n-1} (1 - q^{2s} \bar{b}b) \quad (18.13)$$

The expression  $W_{\mu\lambda}^s$  is of the same form as in the standard theory but the structure coefficients differ from those of the  $SU(2)$  algebra because they depend on  $\bar{b}b$ . Since (18.7) is evaluated on the state  $|0\rangle$  all expressions of the form  $F(\bar{b}b)$  become  $F(|\beta|^2)$ . Therefore the structure constants  $C_{m\ell}^s(\bar{b}b)$  buried in  $W_{\mu\lambda}^s$  and in turn appearing in (18.7) become  $C_{m\ell}^s(|\beta|^2)$ . Then the final reduced form of  $\langle 0|\text{Tr } \mathcal{W}_{\mu\lambda}\mathcal{W}^{\mu\lambda}|0\rangle$  in (18.7) will have one part  $W_{\mu\lambda}^s W_s^{\mu\lambda}$  essentially the same as the standard theory, but with structure constants  $C_{m\ell}^s$  depending on  $|\beta|^2$ . There is also a second part  $\hat{W}_{\mu\lambda}^s \hat{W}_s^{\mu\lambda}$  depending on  $\hat{c}_{m\ell}^s$  which is dependent on  $q$  and  $\beta$ . The sum of these two parts is multiplied by  $\langle 0|\bar{\mathcal{D}}_s \mathcal{D}_s|0\rangle$ , also a function of  $q$  and  $\beta$ . The functions  $c_s(q, \beta)$  will be given in Section 19.

(b) Interactions of Vector Bosons and Fermions.

To describe the boson-fermion interaction we introduce  $\Psi_{Ari}$  where

$$\begin{aligned}\Psi_{1ri} &= \psi_{1r} D_r(a, \bar{a}, b, \bar{b})|i\rangle & A=1 \quad r=\nu, \ell \quad (i=0, 1, 2) \\ \Psi_{2ri} &= \psi_{2r} D_r(a, \bar{a}, b, \bar{b})|i\rangle & A=2 \quad r=u, d \quad (i=0, 1, 2)\end{aligned}$$

Then the boson-fermion interaction is contained in

$$(\bar{\Psi}_A)_{ri} \nabla_{rs} (\Psi_A)_{si'} \quad (18.14)$$

where  $A=1$  labels the  $(\ell, \nu)$  doublet and  $A=2$  labels the quark doublet  $(u, d)$ . To obtain improved agreement with experiment, with the Kobayashi-Maskawa matrix, and with neutrino oscillations it is necessary to introduce flavor states as described in Ref. 19.

In Ref. 16 it is shown that the entire action is invariant under the  $SU(2) \times U(1) \times U_a(1) \times U_b(1)$  symmetries.

## 19 The Higgs Sector in the Knot Model<sup>16</sup>

We follow the standard theory in discussing the vector masses and in the process, we shall determine the constants  $(c_{\pm}, c_3, c_0)$  introduced in (17.6)-(17.8).

The neutral couplings are by (17.5)

$$i(gW_3\tau_3 + g_0W_0\tau_0) \quad (19.1)$$



Introducing the physical fields ( $A$  and  $Z$ ) in the standard way, we have

$$W_0 = A \cos \theta - Z \sin \theta \quad (19.2)$$

$$W_3 = A \sin \theta + Z \cos \theta \quad (19.3)$$

Then (19.1) expressing the neutral couplings becomes

$$i(\mathcal{A}A + \mathcal{Z}Z) \quad (19.4)$$

where

$$\mathcal{A} = g\tau_3 \sin \theta + g_0\tau_0 \cos \theta \quad (19.5)$$

$$\mathcal{Z} = g\tau_3 \cos \theta - g_0\tau_0 \sin \theta \quad (19.6)$$

Now take  $\theta$  to be the Weinberg angle. Then

$$\tan \theta = \frac{g_0}{g} \quad (19.7)$$

and by (19.5) and (19.6)

$$\mathcal{A} = g_0(\tau_3 + \tau_0) \cos \theta \quad (19.8)$$

$$\mathcal{Z} = g(\tau_3 - \tau_0 \tan^2 \theta) \cos \theta \quad (19.9)$$

Let  $|\nu\rangle$  be any neutral state. We shall require

$$(\tau_3 + \tau_0)|\nu\rangle = 0 \quad (19.10)$$

where  $|\nu\rangle$  is a numerically valued two component state. Then by (19.8) and (19.9)

$$\mathcal{A}|\nu\rangle = 0 \quad (19.11)$$

$$\mathcal{Z}|\nu\rangle = \frac{g}{\cos \theta} \tau_3 |\nu\rangle \quad (19.12)$$

Hence the covariant derivative of a neutral state is

$$\nabla = \partial + ig \left[ W_+ \tau_+ + W_- \tau_- + \frac{Z \tau_3}{\cos \theta} \right] \quad (19.13)$$

Denote the neutral Higgs scalar (unitary gauge) by

$$\phi = \rho(x) D_n |0\rangle \quad (19.14)$$

where  $D_n$  is the internal state of the neutral Higgs.

Then the kinetic energy terms of the neutral Higgs in the standard model is by (19.13) and (19.14)

$$\begin{aligned} & \frac{1}{2} \text{Tr}(\overline{\nabla_\mu \varphi} \nabla^\mu \varphi) \\ &= \frac{1}{2} \text{Tr} \langle 0 | \bar{D}_n \\ & \times \left[ \partial_\mu \rho \partial^\mu \rho + g^2 \rho^2 [W_+^\mu W_{+\mu} \bar{\tau}_+ \tau_+ + W_-^\mu W_{-\mu} \bar{\tau}_- \tau_- + \frac{Z^\mu Z_\mu}{\cos^2 \theta} \bar{\tau}_3 \tau_3] \right] D_n | 0 \rangle \end{aligned} \quad (19.15)$$

$$= I \partial_\mu \rho \partial^\mu \rho + g^2 \rho^2 \left[ I_{++} W_+^\mu W_{+\mu} + I_{--} W_-^\mu W_{-\mu} + \frac{I_{33}}{\cos^2 \theta} Z^\mu Z_\mu \right] \quad (19.16)$$

where

$$\begin{aligned} I &= \frac{1}{2} \text{Tr} \langle 0 | \bar{D}_n D_n | 0 \rangle \\ I_{++} &= \frac{1}{2} \text{Tr} \langle 0 | \bar{D}_n \bar{\tau}_+ \tau_+ D_n | 0 \rangle \\ I_{--} &= \frac{1}{2} \text{Tr} \langle 0 | \bar{D}_n \bar{\tau}_- \tau_- D_n | 0 \rangle \\ I_{33} &= \frac{1}{2} \text{Tr} \langle 0 | \bar{D}_n \bar{\tau}_3 \tau_3 D_n | 0 \rangle \end{aligned} \quad (19.17)$$

To agree with the masses predicted by the standard model (19.16) must be reduced to the following

$$\partial_\mu \bar{\rho} \partial^\mu \bar{\rho} + g^2 \bar{\rho}^2 \left[ W_+^\mu W_{+\mu} + W_-^\mu W_{-\mu} + \frac{1}{\cos^2 \theta} Z^\mu Z_\mu \right] \quad (19.18)$$

where

$$\bar{\rho} = I^{1/2} \rho \quad (19.19)$$

To achieve this reduction we impose the following relations

$$\frac{I_{kk}}{I} = 1 \quad k = (+, -, 3) \quad (19.20)$$

or

$$\frac{\text{Tr} \langle 0 | \bar{D}_n (\bar{\tau}_k \tau_k) D_n | 0 \rangle}{\text{Tr} \langle 0 | \bar{D}_n D_n | 0 \rangle} = 1 \quad k = (+, -, 3) \quad (19.21)$$

By (17.6), (17.7) and (19.21) we have

$$|c_k|^{-2} = \frac{\langle 0 | \bar{D}_n (\bar{\mathcal{D}}_k \mathcal{D}_k) D_n | 0 \rangle}{\langle 0 | \bar{D}_n D_n | 0 \rangle} \quad k = (+, -, 3) \quad (19.22)$$

In (17.6) and (17.7) the coefficients  $(c_\pm, c_3)$  were introduced as unknown factors. They are now fixed by (19.22) as definite functions of  $q$  and  $\beta$ . Here the  $\mathcal{D}_k$  are given by (17.9)-(17.11).

The simplest assumption for the neutral Higgs is

$$D_n = D_{00}^0 = 1 \quad (19.23)$$

Then

$$|c_k|^{-2} = \langle 0 | \bar{\mathcal{D}}_k \mathcal{D}_k | 0 \rangle \quad k = (-, +, 3) \quad (19.24)$$

By (17.9)-(17.11)

$$|c_-|^{-2} = \langle 0 | \bar{b}^3 \bar{a}^3 a^3 b^3 | 0 \rangle \quad (19.25)$$

$$= |\beta|^6 \prod_1^3 (1 - q_1^{2t} |\beta|^2) \quad (19.26)$$

$$|c_+|^{-2} = \langle 0 | a^3 b^3 \bar{b}^3 \bar{a}^3 | 0 \rangle \quad (19.27)$$

$$= q^{18} |\beta|^6 \prod_0^2 (1 - q^{2t} |\beta|^2) \quad (19.28)$$

$$|c_3|^{-2} = \langle 0 | \bar{\mathcal{D}}_3 \mathcal{D}_3 | 0 \rangle \quad (19.29)$$

$$= [f(|\beta|^2)]^2 \quad (19.30)$$

If the  $c_k$  satisfy the above relations, then the vector boson masses satisfy the ratios of the standard theory according to (19.18). With the same assumption for the Higgs, we shall next compute the mass ratios in each fermion family.

## 20 The Fermion Mass Term of the Standard Model<sup>10</sup>

In the knot model there is a spectrum of masses that depends on the particular Hamiltonian that is assumed for the knot. We shall restrict this Hamiltonian by the requirements that it lies in the knot algebra and that its eigenstates are  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} |n\rangle$  as we have previously assumed, where  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$  is the kinematic part and the  $|n\rangle$  are the eigenstates of the commuting  $b$  and  $c$ . Finally, in order that the  $H$  introduced in (8.6) qualify as the Hamiltonian of an elementary fermionic knot we shall require that it be compatible with the fermion mass term in the standard theory, namely

$$\mathcal{M} = \bar{L} \varphi R + \bar{R} \varphi L \quad (20.1)$$

where  $L$  and  $R$  are left- and right-chiral Lorentz spinors and  $\varphi$  is the Higgs field, a Lorentz scalar, so that product  $\bar{L} \varphi R$  is Lorentz invariant. In the standard Lagrangian  $L$  and  $\varphi$  are

isotopic doublets.  $(\bar{L}\varphi)$  and  $R$  are separately isotopic singlets and  $\mathcal{M}$  is invariant under the gauged  $SU(2) \times U(1)$  group.

In the knot model  $L$  is additionally a fermionic knot with the charge structure  $D_{-3t_3-3t_0}^{3/2}$ . If a knot singlet is assigned to  $\varphi$ , then  $\varphi$  is neutral (unitary gauge) while the right chiral spinor must have the same knot state as the left chiral spinor, namely,  $D_{\frac{w}{2}\frac{r+1}{2}}^{N/2}$ , in order to preserve the  $U_a(1) \times U_b(1)$  invariance. Then the standard Higgs mechanism is still possible with  $\varphi \sim D_{00}^0$ .

One sees that if the knot state is  $D_{\frac{w}{2}\frac{r+1}{2}}^{3/2}$  for both  $L$  and  $R$ , the relation between  $(t_3, t_0)$  and  $(w, r)$  is different for  $L$  and  $R$ , but the expression for charge, namely  $-\frac{e}{6}(w+r+1)$  is the same for both. In the standard model  $L$  and  $R$  have different relations to the isotopic spin group; here also they have different relations to the isotopic spin, but the same description in the knot algebra.

If  $L$  and  $R$  are now assigned the same internal state, and we treat the mass term in the same way as the other terms of the Lagrangian, then we have

$$L \rightarrow \chi_L(w, r, n) D_{\frac{w}{2}\frac{r+1}{2}}^{3/2} |n\rangle \quad (20.2)$$

$$R \rightarrow \chi_R(w, r, n) D_{\frac{w}{2}\frac{r+1}{2}}^{3/2} |n\rangle \quad (20.3)$$

where  $\chi_L(w, r, n)$  and  $\chi_R(w, r, n)$  are the standard fermionic chiral fields for the particle labelled  $(w, r, n)$ .

Then

$$\mathcal{M}(w, r, n) = \langle n | \bar{D}_{\frac{w}{2}\frac{r+1}{2}}^{3/2} D_{\frac{w}{2}\frac{r+1}{2}}^{3/2} | n \rangle (\bar{\chi}_L \varphi \bar{\chi}_R + \bar{\chi}_R \bar{\varphi} \chi_L) \quad (20.4)$$

By the argument of the standard model

$$\bar{\chi}_L \varphi \chi_R + \bar{\chi}_R \bar{\varphi} \chi_L \quad (20.5)$$

may be reduced to

$$\rho(\bar{\chi}_L \chi_R + \bar{\chi}_R \chi_L) = \rho \bar{\chi} \chi \quad (20.6)$$

where  $\rho$  is the vacuum expectation value of  $\varphi$ , the Higgs field. Then by (20.1)

$$\mathcal{M}(w, r, n) = m(w, r, n) \bar{\chi} \chi \quad (20.7)$$

and by (20.4)

$$m(w, r, n) = \rho(w, r) \langle n | \bar{D}^{3/2}_{\frac{w}{2} \frac{r+1}{2}} D^{3/2}_{\frac{w}{2} \frac{r+1}{2}} | n \rangle \quad (20.8)$$

Then the four spectra (neutrinos, leptons, down quarks, up quarks) may be expressed as follows:

$$m_\nu(n) = \rho(\nu) \langle n | b^3 \cdot \bar{b}^3 | n \rangle \quad (20.9a)$$

$$m_\ell(n) = \rho(\ell) \langle n | \bar{a}^3 \cdot a^3 | n \rangle \quad (20.9b)$$

$$m_d(n) = \rho(d) \langle n | \bar{b}^2 \bar{a} \cdot a b^2 | n \rangle \quad (20.9c)$$

$$m_u(n) = \rho(u) \langle n | a^2 b \cdot \bar{b} \bar{a}^2 | n \rangle \quad (20.9d)$$

where the four prefactors  $(\rho(\nu), \rho(\ell), \rho(d), \rho(u))$  are intended to represent the products of the vacuum expectation value computed at the four local minima in the Higgs potential with the numerical factors in  $D^{3/2}_{mm'}$ . The magnitude of  $\rho$  sets the energy scale and differs for each family. The expressions (20.9) are based on  $SU_q(2)$  as in Table (11.1).

The spectrum of states allowed by the algebra is infinite but there are only three particles in each family. Without additional experimental input we have tentatively assigned these three particles to the states  $n = 0, 1, 2$ , in order of mass where  $n = 0$  corresponds to the lightest particle. The masses in each spectrum are all proportional to the same  $\rho$ ; and hence the mass ratios may be computed without ambiguity in terms of the two parameters ( $q$  and  $\beta$ ) of the model. There are two independent ratios that we choose as

$$M = \frac{m(1)}{m(0)} \quad \text{and} \quad m = \frac{m(2)}{m(1)} \quad (20.10)$$

and that we may express as functions of  $q$  and  $\beta$ . By (20.19a)-(20.19d) one finds for the four families<sup>9,11</sup>

$$\text{neutrinos:} \quad M = m = q^6 \quad (20.11)$$

$$\text{leptons:} \quad \frac{m-1}{m-q^6} = q^3 \frac{M-1}{M-q^6}, \quad |\beta|^2 = q^6 \frac{M-1}{M-q^6} \quad (20.12)$$

$$\text{down quarks:} \quad \frac{m-q^4}{m-q^6} = q^2 \frac{M-q^4}{M-q^6}, \quad |\beta|^2 = \frac{m-q^4}{m-q^6} \quad (20.13)$$

$$\text{up quarks:} \quad \frac{m-q^2}{m-q^6} = q^2 \frac{M-q^2}{M-q^6}, \quad |\beta|^2 = \frac{M-q^2}{M-q^6} \quad (20.14)$$

The empirical input depends on the masses of the elementary fermions. These are well determined for the leptons  $(e, \nu, \tau)$ , but for the quarks they are not even well defined. Since the quarks do not exist as free particles, the quoted masses depend on the theoretical procedure for defining them. There is then a range of “masses” given by the Particle Data Group. Our treatment of mass is limited by its dependence on the mass term of the standard theory, as well as by an arbitrary assignment of  $n$ , and because the binding associated with the gluon and gravitational fields is either ignored or in some indirect way recognized in the mass term.

As already noted, the model permits higher excited states, but no fourth generation particles have been found, and the fourth generation lepton is already excluded by the known width of the  $Z^0$ . Additional physical restrictions on the model are therefore required. It is possible, for example, that these  $q$ -solitons will dissociate into preons at energies below the mass of the fourth excited state. The dissociation energy would depend on the dynamics of the preons.

The neutrino mass spectrum is also a strong constraint on the model; at present the data on this spectrum are compatible with  $q \cong 1$ . In applications to fermionic currents, both in the lepton-neutrino sector and in the Kobayashi-Maskawa sector, the data are compatible with  $q \cong 1$ .

## 21 Gluon Charge<sup>15,17</sup>

The previous considerations are based on electroweak physics. To describe the strong interactions it is necessary according to standard theory to introduce  $SU(3)$  charge. We shall therefore assume that each of the four preon operators appears in triplicate  $(a_i, b_i, c_i, d_i)$  where  $i = R, Y, G$ , without changing the algebra  $(A)$ . We shall assume that these colored preon operators provide a basis for the fundamental representation of  $SU(3)$  just as the colored quark operators do in standard theory, i.e. that color is not an emergent property but appears already at the preon level.

To adapt the electroweak operators to the requirements of the standard theory we make

the following replacements:

$$\text{leptons :} \quad a^3 \rightarrow \epsilon^{ijk} a_i a_j a_k \quad (21.1)$$

$$\text{neutrinos :} \quad c^3 \rightarrow \epsilon^{ijk} c_i c_j c_k \quad (21.2)$$

$$\text{down quarks :} \quad ab^2 \rightarrow a_i (\bar{b}^k b_k) \quad (21.3)$$

$$\text{up quarks :} \quad cd^2 \rightarrow c_i (\bar{d}^k d_k) \quad (21.4)$$

where  $\bar{b}^k$  and  $\bar{d}^k \sim \bar{3}$  representation of  $SU(3)$  and  $(i, j, k) = (R, Y, G)$  and  $(a_i b_i c_i d_i)$  are creation operators for colored preons. Then the creation operators for the leptons and neutrinos are color singlets while the creation operators for the quark states provide a basis for the fundamental representation of  $SU(3)$ , as required by standard theory.

Here  $b$  and  $\bar{b}$ , as well as  $d$  and  $\bar{d}$ , are antiparticles with respect to  $SU(3)$  but have the same values of  $t_3$  and  $t_0$ . Alternatively replace  $\bar{b}^k b_k$  and  $\bar{d}^k d_k$  by  $g^{k\ell} b_k b_\ell$  and  $g^{k\ell} d_k d_\ell$  respectively where  $g^{k\ell}$  is the group metric of  $SU(3)$ .

## 22 The Complementary Models<sup>18</sup>

We have ascribed to the quantum knot the state function  $D_{mm'}^j(q|abcd)$ , an irreducible representation of the knot algebra  $SL_q(2)$ , where the indices  $j = \frac{N}{2}$ ,  $m = \frac{w}{2}$ ,  $m' = \frac{\pm r + 1}{2}$  are restricted to values of  $(N, w, r)$  allowed by the classical (geometrical) knots. The quantum knots have more degrees of freedom than their classical images with the consequence that two quantum knots may be distinguishable when their classical images are topologically indistinguishable.

In particular there are four distinguishable quantum trefoils with  $(w, r) = (\pm 3, \pm 2)$  but only two of their classical images  $(w, r) = (\pm 3, 2)$  are topologically different. In the physical application  $(w, r) = (\pm 3, 2)$  describe the leptons and neutrinos while  $(w, r) = (\pm 3, -2)$  describe the two varieties of quarks, i.e., the two additional quantum knots are required to permit the description of hypercharge and colored fermions.

These considerations have led us to two complementary models of the elementary particles, namely

(a) quantum knots

(b) preon structures

that are the field and particle descriptions of the same particles. The correspondence may be expressed by the following relations according to (14.9) and (14.10)

$$w = n_a - n_d + n_b - n_c (= 2m = -6t_3) \quad (22.1)$$

$$r + 1 = n_a - n_d - n_b + n_c (= 2m' = -6t_0) \quad (22.2)$$

$$N = n_a + n_b + n_c + n_d (= 2j = 6t) \quad (22.3)$$

Here  $(N, w, r)$  describe the number of crossings, the writhe and the rotation of the particle regarded as a quantum knot of field while  $(n_a, n_b, n_c, n_d)$  record the number of  $(a, b, c, d)$  preons in the dual description of the same structure.

We have also described this particle by

$$D_{mm'}^j = D_{-3t_3-3t_0}^{3t} = D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} \quad (22.4)$$

The knot  $(N, w, r)$  and the preon  $(n_a, n_b, n_c, n_d)$  descriptions share the same representation of  $SL_q(2)$  as follows:

In terms of  $(N, w, r)$  one has  $D_{mm'}^j = D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$  where

$$D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}(q|abcd) = \left[ \frac{\langle n'_+ \rangle! \langle n'_- \rangle!}{\langle n_+ \rangle! \langle n_- \rangle!} \right]^{1/2} \sum_{\substack{0 \leq s \leq n_+ \\ 0 \leq t \leq n_-}} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_{q_1} \delta(s+t, n'_+) a^s b^{n_+-s} c^t d^{n_--t} \quad (22.5)$$

and again in terms of  $(N, w, r)$

$$n_{\pm} = \frac{1}{2}[N \pm w] \quad (22.6)$$

$$n'_{\pm} = \frac{1}{2}[N \pm (r+1)] \quad (22.7)$$

The complementary description expressed in terms of the population numbers  $(n_a, n_b, n_c, n_d)$  is

$$D_{mm'}^j = \mathcal{D}_{\nu_a \nu_b}^{N'} \quad (22.8)$$



where  $N'$  is by (14.8) the total number of preons

$$\mathcal{D}_{\nu_a \nu_b}^{N'} = \left[ \frac{\langle n_a + n_c \rangle! \langle n_b + n_d \rangle!}{\langle n_a + n_b \rangle! \langle n_c + n_d \rangle!} \right]^{1/2} \sum_{\substack{N' \geq n_a, n_b \geq 0 \\ N' \geq n_c, n_d \geq 0}} \left\langle \begin{matrix} n_a + n_b \\ n_a \end{matrix} \right\rangle_{q_1} \left\langle \begin{matrix} n_c + n_d \\ n_c \end{matrix} \right\rangle_{q_1} a^{n_a} b^{n_b} c^{n_c} d^{n_d} \quad (22.9)$$

The limits on  $\sum$ , literally translated from  $D_{mm'}^j$  are shown in the expression for  $\mathcal{D}_{\nu_a \nu_b}^{N'}$  but these limits simply describe the requirement that all population numbers,  $n_i$  satisfy  $N' \geq n_i \geq 0$ , since  $N \geq w \geq 0$ .

Here the exponents  $(n_a, n_b, n_c, n_d)$  of  $(abcd)$  are

$$\begin{aligned} n_a &= s & n_b &= n_+ - s \\ n_c &= t & n_d &= n_- - t \end{aligned} \quad (22.10)$$

They are related to  $(n_+, n_-, n'_+, n'_-)$  by

$$\begin{aligned} n_+ &= n_a + n_b & n'_+ &= n_a + n_c \\ n_- &= n_c + n_d & n'_- &= n_b + n_d \end{aligned} \quad (22.11)$$

Since  $a$  and  $d$  have opposite charge and hypercharge, while  $b$  and  $c$  are neutral with opposite hypercharge, we may define the “preon numbers”  $\nu_a$  and  $\nu_b$  as follows

$$\begin{aligned} \nu_a &= n_a - n_d \\ \nu_b &= n_b - n_c \end{aligned} \quad (22.12)$$

The preon numbers are the same for every term of  $\mathcal{D}_{\nu_a \nu_b}^{N'}$ , in (22.9), since

$$\begin{aligned} \nu_a + \nu_b &= 2m = w \\ \nu_a - \nu_b &= 2m' = r + 1 \end{aligned} \quad (22.13)$$

By (22.13) the conservation of the writhe and rotation is equivalent to the conservation of the preon numbers  $\nu_a$  and  $\nu_b$ , and the kinematic factor is described equally well by  $(N, w, r)$  and  $(N, \nu_a, \nu_b)$ . Viewed as twisted loops, the preons could be prevented from unrolling into simple loops by the dynamical conservation of writhe and rotation (computed in the same way as for knots). Viewed as a particle the preon could be conserved by the dynamical conservation of preon numbers.

The trefoil solutions of the equations (22.1)-(22.3) relating  $(N, w, r)$  to  $(n_a, n_b, n_c, n_d)$  are given in Table 22.1:

**Table 22.1**

	$\underline{n_a}$	$\underline{n_b}$	$\underline{n_c}$	$\underline{n_d}$	
$\ell$	3	0	0	0	
$\nu$	0	0	3	0	(22.14)
$d$	1	2	0	0	
$u$	0	0	1	2	

In general

$$\mathcal{D}_{\nu_a \nu_b}^{N'} = \sum_{Nwr} \delta(N', N) \delta(\nu_a + \nu_b, w) \delta(\nu_a - \nu_b, r + 1) D_{\frac{w}{2} \frac{r+1}{2}}^{N/2} \quad (22.15)$$

Since the number of crossings equals the number of preons, one may speculate that there is one preon at each crossing if both preons and crossings are considered pointlike. If the pointlike crossings are labelled  $(\vec{x}_1 \vec{x}_2 \vec{x}_3)$ , then the wave functions of the trefoils representing leptons ( $\ell$ ), neutrinos ( $\nu$ ), down quarks ( $d$ ), up quarks ( $u$ ) are as follows:

$$\Psi_\ell(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \epsilon^{ijk} \psi_i(a|\vec{x}_1) \psi_j(a|\vec{x}_2) \psi_k(a|\vec{x}_3) \quad (22.16)$$

$$\Psi_\nu(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \epsilon^{ijk} \psi_i(c|\vec{x}_1) \psi_j(c|\vec{x}_2) \psi_k(c|\vec{x}_3) \quad (22.17)$$

$$\Psi_d(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \psi_i(a|\vec{x}_1) \bar{\psi}^j(b|\vec{x}_2) \psi_j(b|\vec{x}_3) \quad (22.18)$$

$$\Psi_u(\vec{x}_1 \vec{x}_2 \vec{x}_3) = \psi_i(c|\vec{x}_1) \bar{\psi}^j(d|\vec{x}_2) \psi_j(d|\vec{x}_3) \quad (22.19)$$

where  $i = (R, Y, G)$  and  $\psi_i(a|\vec{x}) \dots \psi_i(d|\vec{x})$  are colored  $\delta$ -like functions localizing the preons at the crossings.

Then the wave function of a lepton describes a singlet trefoil particle containing three preons of charge  $(-e/3)$  and hypercharge  $(-e/6)$ .

The corresponding characterization of a neutrino describes a singlet trefoil containing three neutral preons of hypercharge  $(-e/6)$ .

The wave function of a down quark describes a colored trefoil particle containing one  $a$ -preon with charge  $(-e/3)$  and hypercharge  $(-e/6)$  and two neutral  $b$ -preons with hypercharge  $(e/6)$ .

The corresponding characterization of an up-quark describes a colored trefoil containing two charged  $d$ -preons with charges  $(e/3)$  and hypercharge  $(e/6)$ , and one neutral  $c$ -preon

with hypercharge  $(-e/6)$ .

This hypothetical structure is held together by the trefoil of field connecting the charged preons. A search for this kind of substructure depends critically on the mass of the conjectured preons and the strength with which they are bound. Since there is no empirical information to guide us in discussing the hypothetical preons, either in fixing the masses of the fermionic preons or in determining the fields comprising the binding trefoil, we shall assume that the preonic fermions and bosons conform to the same general rules as the familiar fermions and bosons. Under these assumptions we shall now consider the masses of the fermionic preons and the interactions of the bosonic preons.

## 23 Mass of Preons

Let us assume that the mass of the preon is computed in the same way as we have computed the mass of the elementary fermions, i.e., by adopting the mass terms of the standard theory, namely<sup>1</sup>

$$\mathcal{M} = \bar{L}\varphi R + \bar{R}\varphi L \quad (23.1)$$

where  $L$  and  $R$  are left and right chiral spinors and  $\varphi$  is the Higgs scalar.

We shall assign a  $SU_q(2)$  singlet structure to  $\varphi$  (unitary gauge) and the preon representation  $D_{mm'}^{1/2}$  to both  $L$  and  $R$ . Then we substitute for  $L$  and  $R$  as follows:

$$L \rightarrow \chi_L D_{mm'}^{1/2} |0\rangle \quad (23.2)$$

$$R \rightarrow \chi_R D_{mm'}^{1/2} |0\rangle \quad (23.3)$$

where  $\chi_L$  and  $\chi_R$  are standard fermionic fields and  $D_{mm'}^{1/2} |0\rangle$  describes the internal structure of the preons. Here  $|0\rangle$  is the ground state of the  $SU_q(2)$  algebra. Then

$$\mathcal{M} \equiv \langle 0 | \bar{D}_{mm'}^{1/2} D_{mm'}^{1/2} | 0 \rangle (\bar{\chi}_L \varphi \chi_R + \bar{\chi}_R \varphi \chi_L) \quad (23.4)$$

$$= M(m, m') \bar{\chi} \chi \quad (23.5)$$

where the mass is

$$M(m, m') = \rho(m, m') \langle 0 | \bar{D}_{mm'}^{1/2} D_{mm'}^{1/2} | 0 \rangle \quad (23.6)$$

where  $\rho(m, m')$  is the vacuum expectation value of the Higgs field at a local minimum of the “Higgs potential”.

We shall assume that there are 4 local minima of the Higgs potential, so that

$$\rho(m, m') = \rho\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right) \quad (23.7)$$

For example, the mass of the  $D_{\frac{1}{2}\frac{1}{2}}^{1/2}$  preon is

$$M\left(\frac{1}{2}, \frac{1}{2}\right) = \rho\left(\frac{1}{2}, \frac{1}{2}\right) \langle 0 | \bar{a}a | 0 \rangle \quad (23.8)$$

The mass of the electron computed in the same way is

$$M\left(\frac{3}{2}, \frac{3}{2}\right) = \rho\left(\frac{3}{2}, \frac{3}{2}\right) \langle 0 | \bar{a}^3 a^3 | 0 \rangle \quad (23.9)$$

Then the ratio of the preon mass ( $m_p$ ) to the electron mass ( $m_e$ ) is

$$\frac{m_p}{m_e} = \frac{\rho\left(\frac{1}{2}, \frac{1}{2}\right)}{\rho\left(\frac{3}{2}, \frac{3}{2}\right)} \frac{\langle 0 | \bar{a}a | 0 \rangle}{\langle 0 | \bar{a}^3 a^3 | 0 \rangle} \quad (23.10)$$

The model leads to additional relations between the corresponding Higgs fields.

Let the vacuum expectation values of these fields be  $(\rho_a, \rho_b, \rho_c, \rho_d)$  where

$$(\rho_a, \rho_b, \rho_c, \rho_d) = \left( \rho\left(\frac{1}{2}, \frac{1}{2}\right), \rho\left(\frac{1}{2}, -\frac{1}{2}\right), \rho\left(-\frac{1}{2}, \frac{1}{2}\right), \rho\left(-\frac{1}{2}, -\frac{1}{2}\right) \right) \quad (23.11)$$

respectively. Then under the assumption that the quantum group is  $SU_q(2)$ , one has

$$\begin{aligned} d &= \bar{a} \\ c &= -q_1 \bar{b} \end{aligned} \quad (23.12)$$

and the mass ratio of  $a$  to  $d$  is

$$\frac{m_a}{m_d} = \frac{\rho_a \langle 0 | \bar{a}a | 0 \rangle}{\rho_d \langle 0 | \bar{d}d | 0 \rangle} \quad (23.13)$$

$$= \frac{\rho_a}{\rho_d} \frac{1 - q_1^2 |\beta|^2}{1 - |\beta|^2} \quad (23.14)$$

Similarly the mass ratio of  $b$  to  $c$  is

$$\frac{m_b}{m_c} = \frac{\rho_b \langle 0 | \bar{b}b | 0 \rangle}{\rho_c \langle 0 | \bar{c}c | 0 \rangle} \quad (23.15)$$

$$= q^2 \frac{\rho_b}{\rho_c} \quad (23.16)$$

Since  $a$  and  $d$  and also  $b$  and  $c$  are antiparticles, their masses are equal under the usual forms of the quantum theory that conserve TCP.

Then within the limitations of the model

$$\frac{\rho_a}{\rho_d} = \frac{1 - |\beta|^2}{1 - q_1^2 |\beta|^2} \quad (23.17)$$

$$\frac{\rho_b}{\rho_c} = q_1^2 \quad (23.18)$$

## 24 Electroweak Interactions of Preons

The weak vectors of standard theory belong to the  $j = 3$  and the  $j = 0$  representations of the  $SU_q(2)$  quantum group. Of these only the  $j = 0$  vector interacts with the  $j = 1/2$  preons. The  $j = 1$  vector preon will, however, connect with the  $j = 1/2$  states. This is a new interaction that would contribute to the binding of the preons into a composite particle.

If the internal states are consistently represented by  $D_{-3t_3-3t_0}^{3t}$  the weak vector bosons ( $\vec{W}$ ) of the standard theory correspond to  $j = 3$  and  $t = 1$  as previously shown and given in Table 24.1:

**Table 24.1**

	$t$	$t_3$	$t_0$	$D_{-3t_3-3t_0}^{3t}$
$W^+$	1	1	0	$D_{-30}^3 \sim \bar{b}^3 \bar{a}^3 \equiv \mathcal{D}_+(1)$
$W^-$	1	-1	0	$D_{30}^3 \sim a^3 b^3 \equiv \mathcal{D}_-(1)$
$W^3$	1	0	0	$D_{00}^3 \sim f_3(\bar{b}b) \equiv \mathcal{D}_0(1)$

The corresponding states of the vector preons with  $j = 1$  and  $t = 1/3$  are given in Table 24.2:

**Table 24.2**

	$t$	$t_3$	$t_0$	$D_{-3t_3-3t_0}^{3t}$
$W^+$	$\frac{1}{3}$	$\frac{1}{3}$	0	$D_{-10}^1 = cd \sim \bar{b}\bar{a} \equiv \mathcal{D}_+(1/3)$
$W^-$	$\frac{1}{3}$	$-\frac{1}{3}$	0	$D_{10}^1 = ab \equiv \mathcal{D}_-(1/3)$
$W^3$	$\frac{1}{3}$	0	0	$D_{00}^1 = ad + bc \equiv 1 - (1 + q_1)\bar{b}b = \mathcal{D}_0(1/3)$

The elements of  $D_{-3t_3-3t_0}^{3t}$  representing the vector triplets in both tables may be read as composite creation operators with  $a, b, c, d$  carrying the correct charge and hypercharge for fermionic preons. In both cases the operators  $\mathcal{D}_i(t)$  satisfy the following commutation relations

$$[\mathcal{D}_i(t), \mathcal{D}_j(t)] = c_{ij}^k(t|b\bar{b})\mathcal{D}_k(t) \quad (24.1)$$

$$t = 1, \frac{1}{3} \quad \text{and} \quad (i, j, k) = (+, -, 3)$$

We introduce the covariant derivatives

$$\nabla_\mu(t) = 1 \partial_\mu + \mathcal{W}_\mu(t) \quad (24.2)$$

with the matrix vector potential

$$\mathcal{W}_\mu(t) = W^-(t)t_- \mathcal{D}_-(t) + W^+(t)t_+ \mathcal{D}_+(t) + W^3(t)t_3 \mathcal{D}_3(t) \quad (24.3)$$

where

$$t_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad t_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad t_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (24.4)$$

and

$$[t_i, t_j] = c_{ij}^k t_k \quad (24.5)$$

In the preceding equations (24.3)-(24.5) we have here adopted the same form for the weak preon vectors as for the standard weak vectors. The vector field strengths are

$$\mathcal{W}_{\mu\lambda}(t) = [\nabla_\mu(t), \nabla_\lambda(t)] \quad (24.6)$$

and the electroweak field lagrangian contains the following invariants:

$$L(t) = \text{Tr} \langle 0 | \mathcal{W}_{\mu\lambda}(t) \mathcal{W}^{\mu\lambda}(t) | 0 \rangle \quad t = 1, \frac{1}{3} \quad (24.7)$$

The preon interactions are mediated by the vector fields for which  $t = 1/3$  and not by the vector fields of the standard theory for which  $t = 1$ . These interactions are induced by the kinetic term:

$$\langle 0 | \bar{P} \nabla \left( \frac{1}{3} \right) P | 0 \rangle \quad (24.8)$$

where  $P$  is the doublet  $\begin{pmatrix} a \\ c \end{pmatrix}$  and  $\nabla\left(\frac{1}{3}\right)$  is given by (24.2).

The field lagrangian,  $L(t)$ , reduces to a form similar to that of a lagrangian of a non-Abelian vector field that is derived from a Lie algebra. Unlike the standard case, however,  $L(t)$  has structure constants that depend on  $q$  and  $\beta$ .

## 25 Remarks<sup>18</sup>

The model described in this review is constructed by replacing the point particles of the standard model by quantum knots, described as members of the irreducible representations of  $SL_q(2)$ , the knot algebra. This replacement is carried out by multiplying the normal modes and hence the field operators of the standard model by the state functions of the quantum knots. The essential speculative elements of the modified standard model include (a) the Higgs potential, necessary for standard electroweak theory, and (b) the quantum group  $SL_q(2)$  essential for the conceptual simplification of the electroweak theory that is described here. Both the Higgs potential and  $SL_q(2)$ , raise unanswered questions. Let us consider  $SL_q(2)$  first.

The gauge group of the  $SL_q(2)$  algebra permits a classification of the elementary fermions and hypothetical preons as matrix elements of the  $j = 3/2$  and  $j = 1/2$  representations respectively of either  $SL_q(2)$  or  $SU_q(2)$ . Since these quantum groups describe familiar symmetries when  $q = 1$ , and since only the deviation of  $q$  from unity permits the conceptually simpler picture that the quantum groups permit, the view that one takes of the parameter,  $q$ , becomes important for the view that one takes of our use of  $SL_q(2)$  or  $SU_q(2)$  itself. Like Planck's constant, which normalizes the non-Abelian Heisenberg algebra, the parameter  $q$  also normalizes a non-Abelian algebra but an algebra dependent on  $\epsilon_q$  instead of  $i$ , where  $\epsilon_q$  is also a square root of -1. Unlike  $\hbar$  which has the dimension of an action, the constant  $q$  is dimensionless.

The introduction of substructure, determined by the  $SL_q(2)$  algebra, for the quantum fields in terms of preons resembles the introduction of substructure for the classical fields in terms of field quanta determined by the Heisenberg algebra holding for conjugate field

operators. This analogy suggests a comparison of the  $SL_q(2)$  algebra determined by  $q$  with the Heisenberg algebra determined by  $\hbar$ .

The  $SL_q(2)$  algebra leaves invariant the quadratic form:

$$K = A^t \epsilon_q A \quad (25.1)$$

under the transformations

$$A' = T A \quad T \in SL_q(2) \quad (25.2)$$

where

$$\epsilon_q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad (25.3)$$

as defined in (3.3).

Let us normalize the invariant  $K$  by setting  $K = q^{-1/2}$ . If we now choose

$$A = \begin{pmatrix} D_x \\ x \end{pmatrix} \quad (25.4)$$

we have by (25.1) the  $SL_q(2)$  invariant relation

$$D_x x - q x D_x = 1 \quad (25.5)$$

This equation may be satisfied if  $D_x$  is chosen as the  $q$ -difference operator:

$$D_x \psi(x) \equiv \frac{\psi(qx) - \psi(x)}{qx - x} \quad (25.6)$$

Let

$$P_x = \frac{\hbar}{i} D_x \quad (25.7)$$

Then

$$(P_x x - q x P_x) \psi(x) = \frac{\hbar}{i} \psi(x) \quad (25.8)$$

If  $q$  is near unity

$$q = 1 + \delta \quad (25.9)$$

then

$$D_x \psi(x) = \frac{\psi(x + \delta x) - \psi(x)}{\delta x} \quad (25.10)$$



In the limit  $\delta = 0$

$$D_x \psi(x) = \frac{d}{dx} \psi(x) \quad (25.11)$$

and

$$P = \frac{\hbar}{i} \frac{d}{dx} \quad (25.12)$$

Then (25.8) becomes the Heisenberg commutator

$$(P_x x - x P_x) \psi(x) = \frac{\hbar}{i} \psi(x) \quad (25.13)$$

The operator  $D_x$  may also be expressed in the notation of “basic numbers”, which are useful for discussing  $SL_q(2)$ , as follows: Let

$$\theta = x \frac{d}{dx} \quad (25.14)$$

Then

$$q^\theta f(x) = f(qx) \quad (25.15)$$

and by (25.6)

$$x D_x = \frac{q^\theta - 1}{q - 1} \quad (25.16)$$

or

$$x D_x = \langle \theta \rangle \quad (25.17)$$

$$= \left\langle x \frac{d}{dx} \right\rangle \quad (25.18)$$

so that  $x D_x$  is a “basic dilatation operator”. If  $q$  is near unity,  $D$  resembles the differentiation operator on a lattice space and  $q$  may play the role of a dimensionless regulator.

In view of the physical evidence suggestive of substructure that has been described here, as well as the natural appearance of the non-standard  $q$ -derivatives, it may be possible to utilize  $SL_q(2)$  to describe a finer level of structure than is currently considered.

On the other hand, the basic question is whether  $SL_q(2)$  or  $SU_q(2)$  are fundamental symmetries, and whether  $q$  is accordingly a fundamental physical constant that comes out differently in different contexts, depending on other neglected physics; or whether the  $q$ -model is simply an effective theory, where  $q$  and  $\beta$  are surrogates for physics ignored in the standard model.

Let us finally consider the Higgs potential which has a single minimum in the standard model. The calculation of mass described here is not an essential part of the knot model but presents an interesting possibility if one assumes that the “Higgs potential” has a sequence of local minima associated with one or more scalar fields. Then by consistent use of the Higgs mass term it is in principle possible to calculate all the masses of the model in terms of these minima. The question then would be how these minima should be regarded if they exist. They might be considered simply as given data determining boundary conditions on the model. On the other hand, since the gravitational field has been ignored and since all masses are sources of the gravitational field, there is the possibility that the complete set of “Higgs scalars” might appear as part of the expanded Einstein field, especially since scalar fields appear in supergravity theories.

**Acknowledgement:** I thank J. Smit and A.C. Cadavid for helpful discussions.

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